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A note on certain results involving the polynomials $L_n^{(\alpha,\beta)}(x)$

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Funzioni speciali. — *A note on certain results involving the polynomials* $L_n^{(\alpha,\beta)}(x)$. Nota di REKHA PANDA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota si estendono alcuni recenti risultati di T. R. Prabhakar–Suman Rekha relativi ai polinomi di Konhauser $L_n^{(\alpha,\beta)}(x)$.

1. INTRODUCTION

In their recent paper Rekha and Prabhakar [4] proved the following result:

$$(1.1) \qquad \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu+1)_n \, \Gamma(\alpha n+\beta+1)} L_n^{(\alpha,\beta)}(x) \, {}_1F_1[\mu-\lambda+1 \, ; \mu+n+1 \, ; t] \, t^n$$

$$= \frac{\Gamma(\mu+1)}{\Gamma(\lambda)} e^t \, {}_1F_2^* [(\lambda \, , 1) \, ; (\beta+1 \, , \alpha) \, , (\mu+1 \, , 1) \, ; -xt] \, ,$$

where $L_n^{(\alpha,\beta)}(x)$ is the generalized Konhauser polynomial defined by (see also [2] and [5])

$$(1.2) \qquad L_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha n+\beta+1)}{n!} \sum_{k=0}^{n} \frac{(-n)_k \, x^k}{k! \, \Gamma(\alpha k+\beta+1)} \, , \qquad \mathrm{Re}(\beta) > -1 \, ,$$

α is any complex number with $\mathrm{Re}(\alpha) > 0$, and ${}_pF_q^*[z]$ is Wright's generalized hypergeometric function (cf. [6])

$$(1.3) \qquad {}_pF_q^* \left[\begin{matrix} (a_1 \, , \alpha_1) \, , \cdots , (a_p \, , \alpha_p) \, ; \\ (b_1 \, , \beta_1) \, , \cdots , (b_q \, , \beta_q) \, ; \end{matrix} \; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j+\alpha_j n)}{\prod_{j=1}^{q} \Gamma(b_j+\beta_j n)} \frac{z^n}{n!} \, .$$

Formula (1.1) and its hitherto known special cases were proved by applying certain differential operators including, for instance, the *fractional* derivative operator D_ω^λ defined by

$$(1.4) \qquad D_\omega^\lambda \{\omega^{\mu-1}\} = \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)} \omega^{\mu-\lambda-1} \, , \qquad \lambda \neq \mu \, .$$

(*) Nella seduta del 18 novembre 1977.

In this paper we first present an elementary derivation of the following generalization of (1.1):

$$(1.5)\qquad \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_n}{\Gamma(\alpha n+\beta+1)\prod_{j=1}^{q}(\mu_j)_n} L_n^{(\alpha,\beta)}(x)\, {}_pF_q\left[\begin{matrix}\lambda_1+n,\cdots,\lambda_p+n\,;\\ \mu_1+n,\cdots,\mu_q+n\,;\end{matrix}\; -t\right] t^n$$

$$= \frac{\prod_{j=1}^{q}\Gamma(\mu_j)}{\prod_{j=1}^{p}\Gamma(\lambda_j)}\, {}_pF^*_{q+1}\left[\begin{matrix}(\lambda_1,1),\cdots,(\lambda_p,1)\,;\\ (\beta+1,\alpha),(\mu_1,1),\cdots,(\mu_q,1)\,;\end{matrix}\; -xt\right],$$

where, for convergence, $p \leqq q$ and $|t| < \infty$, or $p = q+1$ and $|t| < 1$.

Evidently, in view of the familiar Kummer's theorem [3, p. 125, Theorem 42]

$$(1.6)\qquad {}_1F_1[a\,;b\,;z] = e^z\,{}_1F_1[b-a\,;b\,;-z]\,,$$

the special case $p = q = 1$ of (1.5) is essentially the same as the Rekha-Prabhakar formula (1.1).

2. Proof of the general result (1.5)

Denoting the first member of (1.5) by $\Delta(x,t)$, if we replace the hypergeometric ${}_pF_q[-t]$ function by its series expansion and collect the powers of t in the resulting double summation, we shall obtain

$$(2.1)\qquad \Delta(x,t) = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_m}{\prod_{j=1}^{q}(\mu_j)_m} \frac{(-t)^m}{m!} \sum_{n=0}^{m} \frac{(-m)_n}{\Gamma(\alpha n+\beta+1)} L_n^{(\alpha,\beta)}(x)\,.$$

By using the definition (1.2), the inner sum in (2.1) can be written in the form

$$\sum_{n=0}^{m} \frac{(-m)_n}{\Gamma(\alpha n+\beta+1)} L_n^{(\alpha,\beta)}(x) = m! \sum_{n=0}^{m}\sum_{k=0}^{n} \frac{(-1)^{n+k}\,x^k}{(m-n)!\,(n-k)!\,k!\,\Gamma(\alpha k+\beta+1)}$$

$$= m! \sum_{k=0}^{m} \frac{x^k}{k!\,(m-k)!\,\Gamma(\alpha k+\beta+1)} \sum_{n=0}^{m-k}(-1)^n \binom{m-k}{n},$$

and since it is easily verified that

$$(2.2)\qquad \sum_{n=0}^{N}(-1)^n\binom{N}{n} = \begin{cases} 1, & \text{if } N = 0,\\ 0, & \text{if } N \geqq 1,\end{cases}$$

we at once have

$$(2.3)\qquad \sum_{n=0}^{m} \frac{(-m)_n}{\Gamma(\alpha n+\beta+1)} L_n^{(\alpha,\beta)}(x) = \frac{x^m}{\Gamma(\alpha m+\beta+1)}, \qquad \forall m \in \{0,1,2,\cdots\}.$$

Now substitute from (2.3) into the right-hand side of (2.1) and interpret the resulting sum by means of (1.3). Thus we arrive at the desired result (1.5) under the conditions stated already.

3. Further generalizations

In order to give a further generalization of our result (1.5), we introduce a general set of polynomials $R_n(x)$ defined by

$$(3.1)\qquad R_n(x) = \sum_{k=0}^{n} \frac{(-n)_k}{k!} c_k x^k, \qquad c_k \neq 0,$$

where $\{c_n\}$ is a sequence of arbitrary complex numbers. Then, as a generalization of (2.3), we have

$$(3.2)\qquad \sum_{n=0}^{m} \frac{(-m)_n}{n!} R_n(x) = c_m x^m, \qquad \forall m \in \{0,1,2,\cdots\},$$

and by applying the proof of (1.5) *mutatis mutandis* we are led to our main result contained in the following

THEOREM. *Corresponding to a given sequence* $\{\gamma_n\}$, *let*

$$(3.3)\qquad \Omega_n(t) = \sum_{r=0}^{\infty} \frac{\gamma_{n+r}\, t^r}{r!}, \qquad |t| < T_0.$$

Also let the polynomials $R_n(x)$ *be defined by* (3.1).

Then

$$(3.4)\qquad \sum_{n=0}^{\infty} R_n(x)\, \Omega_n(-t) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{c_n \gamma_n (-xt)^n}{n!},$$

provided that each side has a meaning.

For $c_n = 1/\Gamma(\alpha n+\beta+1)$, $n \geqq 0$, (1.2) and (3.1) evidently yield the relationship

$$(3.5)\qquad R_n(x) = \frac{n!}{\Gamma(\alpha n+\beta+1)} L_n^{(\alpha,\beta)}(x),$$

and, in view of (3.3), the general formula (3.4) would reduce to our earlier result (1.5) if we further let

$$(3.6)\qquad \gamma_n = \frac{\prod_{j=1}^{p} (\lambda_j)_n}{\prod_{j=1}^{q} (\mu_j)_n}, \qquad \forall n \in \{0, 1, 2, \cdots\}.$$

Yet another interesting special case of (3.4) is the known result [1, p. 134, Eq. (4.9)], which would follow if we set

$$(3.7)\qquad c_n = \frac{\prod_{j=1}^{p-1} (\alpha_j)_n}{(\alpha+1)_n \prod_{j=1}^{q-1} (\beta_j)_n}, \quad \gamma_n = \frac{(\beta+m+1)_n}{(\beta+1)_n}, \qquad \forall m, n \in \{0, 1, 2, \cdots\},$$

and apply Kummer's theorem (1.6) to express the *special* $\Omega_n(-t)$ as a Laguerre polynomial given by

$$(3.8)\qquad L_m^{(\alpha)}(t) = \binom{\alpha+m}{m} {}_1F_1[-m; \alpha+1; t],$$

or equivalently,

$$(3.9)\qquad L_m^{(\alpha)}(t) = \binom{\alpha+m}{m} e^t {}_1F_1[\alpha+m+1; \alpha+1; -t].$$

Finally, we remark that (by appropriately specializing the sequence $\{\gamma_n\}$ and using the definition (1.3) above) we can easily apply our result (3.4) to deduce an interesting extension of (1.5) involving Wright's generalized hypergeometric functions on *both* sides.

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