ATTI ACCADEMIA NAZIONALE DEI LINCEI

CLASSE SCIENZE FISICHE MATEMATICHE NATURALI

RENDICONTI

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Which operators generate cosine operator functions?

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **63** (1977), n.5, p. 314–317.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1977_8_63_5_314_0>

Analisi matematica. — *Which operators generate cosine operator functions?* Nota di DIETER LUTZ, presentata (*) dal Socio E. MARTINELLI a nome del compianto Socio B. SEGRE.

RIASSUNTO. — All'importante questione posta nel titolo viene qui data risposta per operatori lineari normali, assegnando una semplice condizione necessaria e sufficiente che fa intervenire soltanto la forma dello spettro.

Let X denote a complex Banach space, and let B (X) be the algebra of bounded linear operators on X. $\sigma(A)$, $\rho(A)$, and $R(z, A) = (zI - A)^{-1}$ stand for the spectrum, the resolvent set, and the resolvent operator of an operator A on X.

A cosine operator function C on X is a strongly continuous mapping $C : \mathbf{R} \to B(X)$, satisfying $C(0) = I$ (the identity operator) and

$$C(s+t) + C(s-t) = 2\,C(t)\,C(s)$$

for $s, t \in \mathbf{R}$.

If C is a cosine operator function, its infinitesimal generator A is defined by

$$A = C''(0)$$

with domain $D(A) = \{x \in X, C(\cdot)\,x \in C^2(\mathbf{R}, X)\}$. Then A is a densely defined closed linear operator on X. In addition, there is a $w \geq 0$ such that for every $z \in \mathbb{C}$ with $\operatorname{Re} z > w$ we have $z^2 \in \rho(A)$ and for every $x \in X$

$$zR(z^2, A)\,x = \int_0^\infty e^{-zt}\,C(t)\,x\,dt$$

(for these and the following basic facts see [5], [1], [3]).

For a densely defined closed linear operator A on X to be the infinitesimal generator of a cosine operator function C with

$$\|C(t)\| \leq Me^{wt} \tag{1}$$

(for all $t \in \mathbf{R}$ with constants $M, w \geq 0$) the following two conditions are necessary and sufficient:

i) There is a $w \geq 0$ such that for every $z \in \mathbb{C}$ with $\operatorname{Re} z > w$ $z^2 \in \rho(A)$ follows, and

(*) Nella seduta del 18 novembre 1977.

ii) for all $n \geq 0$ and every $z \in \mathbb{C}$ with $\operatorname{Re} z > w$ we have

$$\left\| \frac{d^n}{dz^n} zR(z^2, A) \right\| \leq \frac{M \cdot n!}{2} \left(\frac{1}{(\operatorname{Re} z - w)^{n+1}} + \frac{1}{(\operatorname{Re} z + w)^{n+1}} \right). \tag{2}$$

We remark that the some what simpler condition

$$\left\| \frac{d^n}{dz^n} zR(z^2, A) \right\| \leq \frac{M' \cdot n!}{(\operatorname{Re} z - w)^{n+1}} \tag{2'}$$

which one can find in [1], is also (together with *i*)) necessary and sufficient for A to generate a cosine operator function, but it does not give the correct constant M in (1).

Though Fattorini in [3] gave a thorough study of second order differential equations using cosine operator functions (instead of employing the more familiar method of reducing higher order equations to first order systems and then using semigroups), concrete applications have been severely restricted by the impossibility to verify (2) in the examples given (cf. the remarks of Sova in [5], p. 36). Contrary to the situation for semigroups no simple sufficient criterion could be found to substitute the condition ii).

The aim of this note is to show that for a large class of examples (in fact, it contains the most important second order differential operators acting on Hilbert space) condition ii) is fulfilled if only i) is true.

Let X be a Hilbert space and let A denote a closed normal operator on X with spectral measure E. We assume that i) is true, that is, there is a $w \geq 0$ with $\sigma(A) \subset \mathbb{C} - \{z^2 \mid \operatorname{Re} z > w\}$.

For $z \in \mathbb{C}$ with $\operatorname{Re} z > w$ we have

$$zR(z^2, A) = \int_{\sigma(A)} \frac{z}{z^2 - u} \, dE(u).$$

and consequently

$$\frac{d^n}{dz^n} zR(z^2, A) = \int_{\sigma(A)} \frac{d^n}{dz^n} \frac{z}{z^2 - u} \, dE(u).$$

Now take $u \in \sigma(A)$, then $\operatorname{Re} \sqrt{u} \leq w$ (independently of the root chosen) and so

$$\cosh \sqrt{u}\, t \leq e^{\sqrt{u}\, t} \leq e^{(\operatorname{Re} \sqrt{u})t} \leq e^{wt}$$

for all $t \in \mathbf{R}$ and uniformly for all $u \in \sigma(A)$.

Here we have used the function

$$\cosh \sqrt{a}\, t := \sum_{i=0}^{\infty} \frac{a^i t^{2i}}{(2i)!},$$

where the root in the argument is used in a purely symbolic manner.

Now we can apply Sova's criterion (2) to the scalar valued cosine function $t \to \cosh \sqrt{u}\, t$, which has the infinitesimal generator u. Consequently,

$$\left| \frac{d^n}{dz^n} \frac{z}{z^2 - u} \right| \leq \frac{n!}{2} \left(\frac{1}{(\operatorname{Re} z - w)^{n+1}} + \frac{1}{(\operatorname{Re} z + w)^{n+1}} \right) \tag{3}$$

for all $u \in \sigma(A)$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > w$. Then, the functional calculus for unbounded normal operators provides us with

$$\left\| \frac{d^n}{dz^n} z R(z^2, A) \right\| \leq \left\| \int_{\sigma(A)} \frac{d^n}{dz^n} \frac{z}{z^2 - u} \, dE(u) \right\| = \sup_{u \in \sigma(A)} \left| \frac{d^n}{dz^n} \frac{z}{z^2 - u} \right|$$

$$\leq \frac{n!}{2} \left(\frac{1}{(\operatorname{Re} z - w)^{n+1}} + \frac{1}{(\operatorname{Re} z + w)^{n+1}} \right)$$

and we have achieved

THEOREM 1. *A (unbounded) normal operator* A *on the Hilbert space* X *is the infinitesimal generator of a cosine operator function* C *if and only if there is a* $w \geq 0$ *such that* $z^2 \in \rho(A)$ *for every* $z \in \mathbb{C}$ *with* $\operatorname{Re} z > w$. *In this case* $\| C(t) \| \leq e^{wt}$ *for every* $t \in \mathbf{R}$.

It is proven in [4] that the infinitesimal generator of a uniformly bounded cosine operator function C (that is, $\| C(t) \| \leq M$ for all $t \in \mathbf{R}$ and some $M \geq 0$) has nonpositive real spectrum.

COROLLARY 2. *A normal operator* A *on the Hilbert space* X *is the infinitesimal generator of a uniformly bounded cosine operator function* C *if and only if* A *is selfadjoint with nonpositive spectrum. In this case,* $\| C(t) \| \leq 1$ *for all* $t \in \mathbf{R}$.

Now let X be a Banach space, and let A denote a scalar spectral operatoı in thę sense of Dunford-Schwartz [2] with spectral measure E. Then obviously the statements of Theorem 1 and Cororollary 2 remain true with one exception: There is a $M \geq 1$ such that for bounded continuous functions on $\sigma(A)$ we have

$$\left\| \int_{\sigma(A)} f(u) \, dE(u) \right\| \leq M \cdot \sup_{t \in \sigma(A)} | f(u) | , \tag{4}$$

but in general this inequality is not true with $M = 1$. So the last sentence in Theorem 1 and Cororollary 2 has to be given the following form: In this case $\| C(t) \| \leq M \cdot e^{wt}$ (or $\leq M$) for all $t \in \mathbf{R}$, where M denotes the minimal $M \geq 1$ for which (4) is true.

Clearly, we could have given a proof of Theorem 1 without applying Sova's criterion (2) to the scalar valued cosine functions. One can write

$$\frac{d^n}{dz^n} z (z^2 - u)^{-1} = p_n(z) (z^2 - u)^{-(n+1)}$$

where

$$p_0(z) = z$$

$$p_{n+1}(z) = p_n'(z)(z^2 - u) - 2(n+1) z p_n(z)$$

and verify the boundedness condition (3) directly.

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