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Common fixed points on complete metric spaces

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Analisi matematica. — *Common fixed points on complete metric spaces.* Nota di BRIAN FISHER, presentata (*) dal Socio E. MARTINELLI a nome del compianto Socio B. SEGRE.

RIASSUNTO. — Si dimostra che, se S e T sono applicazioni di uno spazio metrico completo X in sè, con T continua, tale che

$$\rho(STx, TSy) \leq c \max\{\rho(Tx, Sy), \rho(x, y)\}$$

per tutti gli x, y di X, dove $0 \leq c < 1$, allora S ed T hanno un unico punto fisso comune.

The following theorem was proved in a paper by Ray, see [2]:

THEOREM 1. *If* S *and* T *are two mappings of the metric space* X *into itself such that*

$$\rho(Sx, Ty) \leq c\rho(x, y)$$

for all x, y *in* X, *where* $0 \leq c < 1$ *and if for some* x_0 *in* X *the sequence* $\{x_n\}$ *consisting of the points*

$$x_{2n+1} = Sx_{2n} \quad , \quad x_{2(n+1)} = Tx_{2n+1} , \quad n = 0, 1, 2, \cdots,$$

has a subsequence $\{x_{n_k}\}$ *convergent to a point* z *in* X, *then* S *and* T *have the unique common fixed point* z.

It was shown in [1] that this theorem is an immediate consequence of the following theorem:

THEOREM 2. *If* S *and* T *are two mappings of the metric space* X *into itself such that*

$$\rho(Sx, Ty) \leq c\rho(x, y)$$

for all x, y *in* X, *where* $0 \leq c < 1$, *then* S *and* T *are identical contraction mappings.*

We now prove a theorem for two mappings S and T which are not necessarily identical.

THEOREM 3. *If* S *is a mapping and* T *is a continuous mapping of the complete metric space* X *into itself such that*

$$\rho(STx, TSy) \leq c \max\{\rho(Tx, Sy), \rho(x, y)\}$$

for all x, y *in* X, *where* $0 \leq c < 1$, *then* S *and* T *have a unique common fixed point* z.

(*) Nella seduta del 18 novembre 1977.

Proof. Let x be an arbitrary point in X. Then

$$\rho((ST)^n x, T(ST)^n x) \leq c \max\{\rho(T(ST)^{n-1} x, (ST)^n x), \rho((ST)^{n-1} x, T(ST)^{n-1} x)\}$$
$$\leq c \max\{c\rho((ST)^{n-1} x, T(ST)^{n-1} x), c\rho(T(ST)^{n-2} x, (ST)^{n-1} x), \rho((ST)^{n-1} x, T(ST)^{n-1} x)\}$$
$$= c \max\{\rho((ST)^{n-1} x, T(ST)^{n-1} x), c\rho(T(ST)^{n-2} x, (ST)^{n-1} x\}$$
$$\leq c^2 \max\{\rho(T(ST)^{n-2} x, (ST)^{n-1} x), \rho((ST)^{n-2} x, T(ST)^{n-2} x)\}$$
$$\leq c^n \max\{\rho(Tx, STx), \rho(x, Tx)\}.$$

Similarly, we have

$$\rho(T(ST)^n x, (ST)^{n+1} x) \leq c^n \max\{\rho(STx, TSTx), \rho(Tx, STx)\}.$$

Since $c < 1$, it follows that the sequence

$$\{x, Tx, STx, \cdots, (ST)^n x, T(ST)^n x, \cdots\}$$

is a Cauchy sequence in the complete metric space X and so has a limit z in X. Thus

$$\lim_{n\to\infty} (ST)^n x = \lim_{n\to\infty} T(ST)^n x = z$$

and since T is continuous it follows that $Tz = z$ so that z is a fixed point of T.

We now have

$$\rho(T(ST)^n x, Sz) = \rho(T(ST)^n x, STz)$$
$$\leq c \max\{\rho((ST)^n x, Tz), \rho(T(ST)^{n-1} x, z)\}$$

and on letting n tend to infinity it follows that

$$\rho(z, Sz) \leq c \max\{\rho(z, Tz), \rho(z, z)\} = 0.$$

Thus $Sz = z$ and so z is a common fixed point of S and T.

Now suppose that there exists a second common fixed point z'. Then

$$\rho(z, z') = \rho(STz, TSz')$$
$$\leq c \max\{\rho(Tz, Sz'), \rho(z, z')\}$$
$$= c\rho(z, z')$$

and, since $c < 1$, it follows that $z = z'$ and so the common fixed point of S and T is unique. This completes the proof of the theorem.

We now note that the mappings S and T in Theorem 1 are not necessarily equal. This is easily seen by considering a complete metric space X containing at least two points. Define a continuous mapping T on X by

$$Tx = x$$

for all x in X and define a mapping S on X by

$$Sx = z$$

for all x in X, where z is a fixed point in X. Then

$$STx = TSx = z$$

for all x in X and so the conditions of the theorem are satisfied with $c = \frac{1}{2}$, but S is not equal to T.

This example also shows that the mappings S and T can possibly have other fixed points although a common fixed point has to be unique.

The condition that T has to be continuous is also necessary. This can be seen by letting X be the set of real numbers x with $0 \leq x \leq 1$. Define a metric by

$$\rho(x, y) = |x - y|$$

for all x, y in X and define discontinuous mappings S = T on X by

$$T(0) = 1 \quad , \quad Tx = \tfrac{1}{2} x , \qquad \text{for} \quad x \neq 0 .$$

X is complete and

$$\rho(STx, TSy) \leq \tfrac{1}{2} \max \{\rho(Tx, Sy), \rho(x, y)\}$$

for all x, y in X but S and T have no fixed point.

By noting that

$$b\rho(Tx, Sy) + c\rho(x, y) \leq \max \{\rho(Tx, Sy), \rho(x, y)\}$$

where $0 \leq b, c, b + c \leq 1$, we have the following theorem:

THEOREM 4. *If* S *is a mapping and* T *is a continuous mapping of the complete metric space* X *into itself such that*

$$\rho(STx, TSy) \leq b\rho(Tx, Sy) + c\rho(x, y)$$

for all x, y *in* X, *where* $0 \leq b, c, b + c < 1$, *then* S *and* T *have a unique common fixed point* z.

On putting S = T in Theorem 3 and Theorem 4 we have the following two theorems:

THEOREM 5. *If* T *is a continuous mapping of the complete metric space* X *into itself such that*

$$\rho(T^2 x, T^2 y) \leq c \max \{\rho(Tx, Ty), \rho(x, y)\}$$

for all x, y *in* X, *where* $0 \leq c < 1$, *then* T *has a unique fixed point* z.

THEOREM 6. *If* T *is a continuous mapping of the complete metric space* X *into itself such that*

$$\rho(T^2 x, T^2 y) \leq b\rho(Tx, Ty) + c\rho(x, y)$$

for all x, y *in* X, *where* $0 \leq b, c, b + c < 1$, *then* T *has a unique fixed point* z.

The last example shows that the condition that T be continuous in these two theorems is still necessary.

In the final two theorems the two mappings S and T can both be discontinuous. First of all we have

THEOREM 7. *If* S *and* T *are mappings of the metric space* X *into itself such that*

$$\rho(STx, TSy) \leq c \max\{\rho(Tx, Sy), \rho(x, y)\}$$

for all x, y *in* X, *where* $0 \leq c < 1$ *and if* $Sx = Tx$ *for some* x *in* X, *then* $S^n x = T^n x$ *for* $n = 1, 2, \cdots$

Proof. Suppose that $Sx = Tx$ for some x in X. Assuming that $S^r x = T^r x$ for $r = 1, 2, \cdots, n$ and some n, we have

$$\rho(ST^n x, TS^n x) \leq c \max\{\rho(T^n x, S^n x), \rho(T^{n-1} x, S^{n-1} x)\} = 0$$

by our assumption. It follows that

$$ST^n x = TS^n x$$

or

$$S^{n+1} x = T^{n+1} x$$

since $S^n x = T^n x$ by our assumption. The result now follows by induction.

Finally we have

THEOREM 8. *If* S *and* T *are mappings of the metric space* X *into itself such that*

$$\rho(STx, TSy) \leq b\rho(Tx, Sy) + c\rho(x, y)$$

for all x, y *in* X, *where* $0 \leq b, c, b + c < 1$ *and if* $Sx = Tx$ *for some* x *in* X, *then* $S^n x = T^n x$ *for* $n = 1, 2, \cdots$

REFERENCES

[1] B. FISHER – *On contraction mappings*, «Colloq. Math.», to appear.
[2] B. K. RAY (1976) – *Contraction mappings and fixed points*, «Colloq. Math.», *35*, 223–234.