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Francesco Mercuri

On the rational cohomology of the spaces of unparametrized closed curves

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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On the rational cohomology of the spaces of unparametrized closed curves.* Nota di Francesco Mercuri (*), presentata (**) dal Socio G. Zappa.

Riassunto. — Sia M una varietà liscia e chiusa contenuta in uno spazio euclideo $\mathbf{R}^k$. Siano poi $H^1(S^1, \mathbf{R}^k)$ (S^1 = circonferenza) lo spazio di Sobolev delle curve assolutamente continue $c: S^1 \to \mathbf{R}^k$ tali che $\|\dot{c}(t)\| \in L^2(S^1)$. Sia ΛM la sottovarietà di $H^1(S^1, \mathbf{R}^k)$ delle curve di M. Su ΛM opera S^1 per rotazioni; sia ΠM lo spazio quoziente. In questa Nota si studia la coomologia razionale di ΠM; più precisamente posto $\Pi^0 M = \Lambda^0 M = M$ per il sottospazio delle curve costanti si studia la struttura dell'accoppiamento (« cap » prodotto):

$$H^*(\Pi M, \Pi^0 M) \otimes H_*(\Pi M - \Pi^0 M) \xrightarrow{\cap} H^*(\Pi M, \Pi^0 M)$$

Introduction

Let M be a closed smooth manifold that we think of, for simplicity, as being embedded in some euclidean space $\mathbf{R}^k$.

Let $S^1 = \mathbf{R}/\mathbf{Z}$ be the unit circle parametrized between 0 and 1.

Let $H^1(S^1, \mathbf{R}^k)$ be the Sobolev space of absolutely continuous curves $c: S^1 \to \mathbf{R}^k$ such that $\|\dot{c}(t)\| \in L^2(S^1)$, with the standard norm, and $\Lambda M = \{c \in H^1(S^1, \mathbf{R}^k) : c(t) \in M, \forall t \in S^1\}$. Then ΛM has the structure of a smooth riemannian submanifold of $H^1(S^1, \mathbf{R}^k)$ (strictly infinite-dimensional if $\dim(M) = n > 0$). If we indicate by $C^0(S^1, M)$ the space of continuous maps from S^1 to M with the compact-open topology, the inclusion $\Lambda M \to C^0(S^1, M)$ is a homotopy equivalence (see [9]).

(*) Part of this work was done while the Author was an SFB-40 guest at the University of Bonn.

(**) Nella seduta del 18 novembre 1977.

S^1 acts continuously on ΛM by rotations, and we denote by ΠM the quotient space.

The spaces ΛM and ΠM have been studied for a long time, especially in order to prove the existence of closed extremals of variational problems via critical point theory.

M. Morse noticed that for a variational problem $\tilde{L} : \Lambda M \to \mathbf{R}$ invariant under the action of S^1, the now standard methods of critical point theory can be extended to the induced function $\tilde{L}_\Pi : \Pi M \to \mathbf{R}$, with the advantage that the topology on ΠM is richer than the one ΛM (i.e., the invariance of $\tilde{L}$ under S^1 forces $\tilde{L}$ to have more critical points). We notice that the action of S^1 is not always smooth and ΠM is not necessarily a manifold.

The purpose of this paper is to study the rational cohomology of ΠM. More precisely, if we indicate by $\Pi^0 M = \Lambda^0 M \simeq M$ the subspace of constant curves, there is a cap product pairing (see [6])

$$H^*(\Pi M , \Pi^0 M) \otimes H_*(\Pi M - \Pi^0 M) \xrightarrow{\cap} H^*(\Pi M , \Pi^0 M) ,$$

and we will be interested in the structure of this pairing.

If $z_1 , z_2 \in H^*(\Pi M , \Pi^0 M)$ and $z_1 = z_2 \cap \zeta$ with $\zeta \in H_*(\Pi M , \Pi^0 M)$ $(* > 0)$, we will say that z_1 and z_2 are subordinated. We shall prove the existence of at least two such classes for any compact simple-connected M, and the existence of infinitely many such classes for a large class of manifolds, and we shall give applications to the existence of closed extremals for certain variational problems.

We want to thank W. Klingenberg and R. K. Lashof for helpful conversations and encouragement.

Notations and known facts

Let X be a countable CW-complex with nilpotent fundamental group. A minimal model for X is a graded differential algebra over the rationals (or reals) which is graded commutative and associative,

$$\mathcal{M}(X) = \{\mathcal{M}_k(X) , d_k : \mathcal{M}_k(X) \to \mathcal{M}_{k+1}(X)\}_{k \geq 0} ,$$

such that:

1) $\mathcal{M}(X)$ is free except for the associativity and graded commutativity relations;

2) The subspace of $\mathcal{M}_k(X)$ spanned by the k-dimensional generators of $\mathcal{M}(X)$ is isomorphic to $(\pi_k(X) \otimes Q)^*$;

3) The differential of a k-dimensional generator is either zero or a polynomial in lower degree generators;

4) $H^*(\mathcal{M}(X)) \simeq H^*(X ; Q)$.

Thus a minimal model is a tensor product of polynomial algebras on even-dimensional generators and exterior algebras on odd-dimensional generators. Such minimal models exist and are unique up to isomorphism (see [6] and [11]).

For the rest of this paper, M will be a closed, simply-connected manifold, and homology and cohomology will always be with rational coefficients.

The following is an easy consequence of the existence of minimal models and the fact that the rational cohomology of an H-space is a free algebra generated by its rational homotopy:

1. LEMMA. (a) $\pi_*(M) \otimes Q$ *is non-trivial in some odd dimension*;

(b) $H^*(\Omega M ; Q)$ *contains a polynomial sub-algebra.*

Given a minimal model for M, $\mathcal{M}(M)$, a minimal model for ΛM can be constructed as follows (see [12]):

Since the fibration $\Omega M \to \Lambda M \to M$ has a section, $\pi_*(\Lambda M) \simeq \simeq \pi_*(M) \oplus \pi_*(\Omega M)$. So, for any generator $y \in \mathcal{M}_k(M)$ we have two generators $y \in \mathcal{M}_k(\Lambda M)$ and $\bar{y} \in \mathcal{M}_{k-1}(\Lambda M)$. This defines $\mathcal{M}(\Lambda M)$ as a graded commutative and associative free algebra. We extend the " bar operator " to all $\mathcal{M}(\Lambda M)$ as a derivation, that is,

$$\overline{xy} = \bar{x}y + (-1)^{|x|} x\bar{y} \qquad (|x| = \text{degree of } x)$$

and we define the diferential $d_\Lambda : \mathcal{M}_*(\Lambda M) \to \mathcal{M}_{*+1}(\Lambda M)$ by

$$d_\Lambda y = dy \qquad \text{on the unbarred generators}$$

and

$$d_\Lambda \bar{y} = -\overline{dy} \qquad \text{on the barred generators.}$$

Using this model, Sullivan (see [12]) constructed many classes in $H^*(\Lambda M)$ which restrict non-trivially to classes in $H^*(\Omega M)$. Naturally, as soon as one of those classes is even-dimensional it generates a polynomial algebra inside $H^*(\Lambda M)$. As a consequence of the existence of those classes Sullivan proved that the sequence of betti numbers of ΛM is unbounded if and only if $H^*(M)$ is not generated by only one element. This last fact is of great importance in the theory of closed geodesics, since a theorem of Gromoll and Meyer ensures the existence of infinitely many closed geodesics for any riemannian metric on a closed manifold M such that the sequence of betti numbers of ΛM is unbounded (notice that the last condition depends only on the homotopy type of M).

The following is again an immediate consequence of Sullivan's construction:

2. LEMMA. *If one the following two conditions is satisfied, then* $H^*(\Lambda M)$ *contains a polynomial subalgebra*:

(a) *the first non-vanishing rational homotopy group of* M *is odd-dimensional*;

(b) M *has the homotopy type of the product of two closed manifolds (of positive dimension).*

Unfortunately, is some interesting cases $H^*(\Lambda M)$ is not (multiplicatively) rich enough. In fact, if $H^*(M) \simeq Q[x]/(x^{n+1})$ with $x \in H^{2k}(M)$, then $H^*(\Lambda M)$ is the tensor product of a polynomial algebra on a $2(k(n+1)-1)$-dimensional generator and the ring $\Lambda^+(x, \bar{x})/(x^{n+1}, x^n \bar{x})$, where $\Lambda^+(x, \bar{x})$ is ideal in the free algebra on x and $\bar{x}$ of the elements of positive degree. Notice that in this case the product of $n+1$ elements of positive degree in $H^*(\Lambda M)$ is always zero (see [12]).

This justifies the idea of looking for subordinated cohomology classes in ΠM.

Remark. Since ΛM is a smooth Hilbert (stricly infinite-dimensional) manifold and $M \simeq \Lambda^0 M$ a compact submanifold, the inclusion $\Lambda M - \Lambda^0 M \to \Lambda M$ is a homotopy equivalence (see [3]), and therefore the existence of a polynomial algebra in $H^*(\Lambda M)$ implies the existence of infinitely many subordinated cohomology classes.

The cohomology of ΠM

If G is a topological group acting on a space X and $G \to EG \to BG$ is the universal bundle, we will denote by X_G the balanced product $EG \times_G X$. We have the diagram

$$\begin{array}{ccc} X & & \\ \downarrow & & \\ X_G & \xrightarrow{\pi_2} & X/G \\ \pi_1 \downarrow & & \\ BG & & \end{array}$$

where π_1 and π_2 are the obvious maps. Moreover, π_1 is a fibre bundle and, if $[y] \in X/G$ is the orbit of $y \in X$ under G, $\pi_2^{-1}([y]) \simeq B(G_y)$, where G_y is the isotropy subgroup of y (see [2]).

3. Lemma. $\pi_2^*: H^*((\Lambda M - \Lambda^0 M)/S^1) \to H^*((\Lambda M - \Lambda^0 M)/S^1)$ *is an isomorphism.*

Proof. Since S^1 acts without fixed points on $\Lambda M - \Lambda^0 M$, all the isotropy subgroups are finite, so, for $[y] \in (\Lambda M - \Lambda^0 M)/S^1$, $\pi_2^{-1}([y])$ is rationally acyclic. Since S^1 is compact, the projection map

$$\pi: (\Lambda M - \Lambda^0 M) \to (\Pi M - \Pi M) = (\Lambda M - \Lambda^0 M)/S^1$$

is a closed. For any n, consider the n-universal bundle $S^1 \to E_n S^1 \to B_n S^1$ and the relative situation

$$\pi_{2,n}: E_n S^1 \times_{S^1} (\Lambda M - \Lambda^0 M) \to (\Pi^0 M - \Pi^0 M).$$

Since $E_n S^1$ is compact and π is closed, it follows that $\pi_{2,n}$ is closed and, for all $[y] \in (\Pi M - \Pi^0 M)$, $H^*(\pi_{2,n}^{-1}([y])) = 0$ for $* < n$. It follows then, from the Vietoris mapping Theorem [1] (see [10]), that $\pi_{2,n}^*$ is an isomorphism for $* < n$ and, therefore, letting n grow, π_2^* is an isomorphism.

4. LEMMA. *The inclusion* $i_{S^1} : (\Lambda M - \Lambda^0 M)_{S^1} \to \Lambda M_{S^1}$ *is a homotopy equivalence.*

Proof. As already remarked, the inclusion $\Lambda M - \Lambda^0 M \to \Lambda M$ is a homotopy equivalence. The lemma follows looking at the bundle map

$$\begin{array}{ccccc} (\Lambda M - \Lambda^0 M) & \longrightarrow & (\Lambda M - \Lambda^0 M)_{S^1} & \longrightarrow & BS^1 \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_{S^1}} & & \downarrow{\scriptstyle 1} \\ \Lambda M & \longrightarrow & \Lambda M_{S^1} & \longrightarrow & BS^1 \end{array}$$

We notice, in particular, that, since S^1 acts with fixed points on ΛM, $\pi_2 : \Lambda M_{S^1} \to BS^1$ has a section, and therefore $H^*(\Pi M - \Pi^0 M)$ contains a polynomial algebra generated by $w_2 = \pi_2^*(c_1)$, where c_1 is the generator of $H^*(BS^1)$.

Let $\tilde{E} : \Lambda M \to \mathbf{R}$ be the energy integral

$$\tilde{E}(c) = \frac{1}{2} \int_{S^1} \|\dot{c}\|^2 \, dt .$$

Since $\tilde{E}$ is S^1-invariant, it induces a continuous map

$$\tilde{E}_\Pi : \Pi M \to \mathbf{R} .$$

for $a \in \mathbf{R}$ we set

$$\Lambda^a M = \tilde{E}^{-1}([0, a]) \quad ; \quad \Pi^a M = \tilde{E}_\Pi^{-1}([0, a]) ;$$

$$\Lambda^{a-} M = \tilde{E}^{-1}([0, a)) \quad ; \quad \Pi^{a-} M = \tilde{E}_\Pi^{-1}([0, a)) .$$

$\tilde{E}$ is a C^∞ map and, since the induced riemannian metric on ΛM is S^1-invariant, the vector field $\zeta = -\operatorname{grad} \tilde{E}$ is S^1-equivariant. "Going down" along the integral lines of ζ we can define a semigroup of $\tilde{E}$-decreasing transformations

$$\phi : \ [0, \infty) \times \Lambda M \to \Lambda M ,$$

and, since everything is respected by the S^1-action, a semigroup of $\tilde{E}_\Pi$ decreasing transformations

$$\phi_\Pi : \ [0, \infty) \times \Pi M \to \Pi M .$$

(1) Since ΛM and ΠM are locally contractable, the Alexander cohomology coincides with the singular one.

5. THEOREM. *For* ε *small enough,* ϕ *and* ϕ_Π *define strong deformation retractions*

$$\phi^\varepsilon : \Lambda^\varepsilon M \to \Lambda^0 M \quad \text{and} \quad \phi_\Pi^\varepsilon : \Pi^\varepsilon M \to \Pi^0 M .$$

Proof. see [6].

6. THEOREM. $H^*(\Lambda M_{S^1}, \Lambda^0 M_{S^1}) \simeq H^*(\Pi M, \Pi^0 M)$.

Proof. Using the same arguments as in 3. and 4. applied to the pair $(\Lambda M - \Lambda^0 M, \Lambda^\varepsilon M - \Lambda^0 M)$ (ε small), we have

$$H^*(\Lambda M_{S^1}, \Lambda^\varepsilon M_{S^1}) \simeq H^*(\Pi M - \Pi^0 M, \Pi^\varepsilon M - \Pi^0 M)$$

By 5 we have

$$H^*(\Lambda M_{S^1}, \Lambda^0 M_{S^1}) \simeq H^*(\Lambda M_{S^1}, \Lambda^\varepsilon M_{S^1}),$$

and an excision gives

$$H^*(\Pi M - \Pi^0 M, \Pi^\varepsilon M - \Pi^0 M) \simeq H^*(\Pi M, \Pi^0 M).$$

putting the three isomorphisms together, we have the theorem.

7. COROLLARY. *There is an exact sequence*:

$$\to H^{r+1}(\Lambda M, \Lambda^0 M) \to H^r(\Pi M, \Pi^0 M) \xrightarrow{\cup w_2} H^{r+2}(\Pi M, \Pi^0 M) \to$$
$$\to H^{r+2}(\Lambda M, \Lambda^0 M) \to \cdots$$

with $w_2 \in H^2(\Pi M - \Pi^0 M)$.

Proof. Using 6, this is just the Gysin sequence of the fibration

$$S^1 \to (ES^1 \times \Lambda M, ES^1 \times \Lambda^0 M) \to (\Lambda M_{S^1}, \Lambda^0 M_{S^1}).$$

8. COROLLARY. *The first non-vanishing cohomology groups of* $(\Lambda M, \Lambda^0 M)$ *and* $(\Pi M, \Pi^0 M)$ *occur in the same dimensions and are isomorphic.*

Proof. Immediate from 7.

9. THEOREM. *At least one of the following two assertions holds*:

(1) $H^*(\Lambda M)$ *contains a polynomial algebra*;

(2) $H^*(\Pi M, \Pi^0 M)$ *contains at least two subordinated classes.*

Proof. Consider the minimal models for M and ΛM:

$$0 \to \cdots \to 0 \to \mathcal{M}_{k+1}(M) \to \mathcal{M}_{k+2}(M) \to \cdots$$
$$0 \to \cdots 0 \to \mathcal{M}_k(\Lambda M) \to \mathcal{M}_{k+1}(\Lambda M) \to \mathcal{M}_{k+2}(M) \to \cdots$$

and suppose $\mathcal{M}_{k+1}(M)$ is generated by $x_1, \cdots, x_q$ and $\mathcal{M}_{k+2}(M)$ by $y_1, \cdots, y_p$. We can assume k odd; otherwise, by 3., $H^*(\Lambda M)$ contains a polynomial algebra and the theorem will follow. Let us notice that $\mathcal{M}_{k+1}(M) \simeq H^{k+1}(M)$, $\mathcal{M}_k(\Lambda M) \simeq H^k(\Lambda M)$ and $\mathcal{M}_{k+1}(\Lambda M)$ is generated by $\{x_1, \cdots, x_q, \bar{y}_1, \cdots \bar{y}_p\}$. Now $H^{k+1}(\Lambda M) \simeq \mathcal{M}_{k+1}(\Lambda M)$ because, since $\bar{x}_i \in \mathcal{M}(\Lambda M)$ are exterior elements and k is odd, $d\bar{y}$ cannot be a non-zero polynomial in lower degree generators. But, on the other hand, if $\mathcal{M}_{k+2}(M) \neq 0$, the $\bar{y}_i$ would restrict to non-trivial even-dimensional elements in $H^*(\Omega M)$ and therefore would generate a polynomial algebra in $H^*(\Lambda M)$ and again the theorem follows. If $\mathcal{M}_{k+2}(M) = 0$, looking at the exact sequence of the pair $(\Lambda M, \Lambda^0 M)$ it follows that $H^{k+1}(\Lambda M, \Lambda^0 M) = 0$. From 7 we have

$$0 \to H^k(\Pi M, \Pi^0 M) \simeq H^k(\Lambda M, \Lambda^0 M) \xrightarrow{\cup w_2} H^{k+2}(\Pi M, \Pi^0 M) \to \cdots,$$

and therefore there exist two subordinated classes in $H^*(\Pi M, \Pi^0 M)$.

10. PROPOSITION. $H^*(\Pi^\varepsilon M, \Pi^0 M) \simeq H^*(M) \otimes H^*(BS^1)$.

Proof. By the same argument as in 3,

$$\pi_2^* : H^*(\Pi^\varepsilon M - \Pi^0 M) \to H^*((\Lambda^\varepsilon M - \Lambda^0 M)_{S^1})$$

is an isomorphism. $\Lambda^\varepsilon M$ has the structure of an infinite-dimensional disk bundle over $\Lambda^0 M$ with $\partial\Lambda^\varepsilon M = \Lambda^\varepsilon M - \Lambda^{\varepsilon^-} M$ (see [6]) and S^1-invariant projection $p : \Lambda^\varepsilon M \to \Lambda^0 M$. Therefore we have a bundle map:

$$\begin{array}{ccccc} (\Lambda^\varepsilon M - \Lambda^0 M) & \longrightarrow & (\Lambda^\varepsilon M - \Lambda^0 M)_{S^1} & \longrightarrow & BS^1 \\ \downarrow p & & \downarrow p_{S^1} & & \downarrow 1 \\ \Lambda^0 M & \longrightarrow & \Lambda^0 M & \longrightarrow & BS^1 . \end{array}$$

Since p is a homotopy equivalence, so is p_{S^1}, and the proposition follows.

Looking at the inclusion $(\Pi^\varepsilon M - \Pi^0 M) \to (\Pi M - \Pi^0 M)$ we can construct another exact sequence that will help in an explicit computation of $H^*(\Pi M, \Pi^0 M)$. In fact, the exact sequence of the pair can be written as:

11. $\xrightarrow{\delta} H^r(\Pi M, \Pi^0 M) \to H^r(\Pi M - \Pi^0 M) \to H^r(\Pi^\varepsilon M - \Pi^0 M) \to \cdots$

12. $\xrightarrow{\delta} H^r(\Pi M, \Pi^0 M) \xrightarrow{i^*} H^r(\Lambda M_{S^1}) \xrightarrow{j^*} H^r(\Lambda^0 M_{S^1}) \to \cdots$, where j^* is induced the inclusions $j : \Lambda^0 M_{S^1} \to \Lambda M_{S^1}$.

Now Sullivan has a general formula for computing the minimal model of ΛM_{S^1} (see [11]; really the theorem is stated there for S^1-actions on compact manifolds, but it works in this more general context). Since $\Lambda M \to \Lambda M_{S^1} \to BS^1$ has a cross-section, it follows that the homotopy of ΛM_{S^1} is the direct sum of the homotopies of ΛM and BS^1. So $\mathcal{M}(\Lambda M_{S^1})$ is generated by a two-dimensional class $w_2 \in \pi_2(BS^1) \otimes Q$ and the rational homotopy of ΛM. The differential d' of $\mathcal{M}(\Lambda M_{S^1})$ is given by

13. $$d' w_2 = 0 \quad \text{and}$$
$$d' x = dx + \bar{x} w_2 ,$$

where d is the differential in $\mathcal{M}(\Lambda M)$ and " bar " is the derivation in $\mathcal{M}(\Lambda M)$ defined at the beginning.

In particular, j^* sends the two-dimensional class w_2 in to the corresponding generator of $H^*(\Lambda^0 M_{S^1}) = H^*(M \times BS^1)$, kills the " barred " generators and acts as the identity on the others.

So 12 and 13 give the cohomology of $(\Pi M, \Pi^0 M)$.

Examples and applications

In view of Katok's example of a (non-symmetric) Finsler metric on S^2 with only two closed geodesics, the case in which M has the rational cohomology of a symmetric space of rank one is of great interest.

14. THEOREM. *If* M *is a closed simply-connected manifold with the rational cohomology of a symmetric space of rank one, then there are infinitely many subordinated classes in* $H^*(\Pi M, \Pi^0 M)$.

Proof. For simplicity, we shall do explicit computations for the case $H^*(M) \simeq H^*(S^n)$, $n = 2k$, the other cases being similar. An easy computation shows that the minimal model of $(\Lambda S^n)_{S^1}$ is generated by:

$$
\begin{aligned}
&w_2 \in \mathcal{M}_2((\Lambda S^n)_{S^1}) && \text{with} \quad d' w_2 = 0 ;\\
&\bar{x} \in \mathcal{M}_{n-1}((\Lambda S^n)_{S^1}) && \text{with} \quad d' \bar{x} = 0 ;\\
&x \in \mathcal{M}_n((\Lambda S^n)_{S^1}) && \text{with} \quad d' x = \bar{x} w_2 ;\\
&\bar{y} \in \mathcal{M}_{2(n-1)}((\Lambda S^n)_{S^1}) && \text{with} \quad d' \bar{y} = x\bar{x} ;\\
&y \in \mathcal{M}_{2n-1}((\Lambda S^n)_{S^1}) && \text{with} \quad d' y = x^2 + \bar{y} w_2 .
\end{aligned}
$$

Then $H^*(\Pi S^n - \Pi^0 S^n)$ is generated by $w_2 \in H^2(\Pi S^n - \Pi^0 S^n)$ and $\bar{x} \in H^{n-1}(\Pi S^n - \Pi^0 S^n)$ with the relation $\bar{x} w_2 = 0$ and contains elements of the type $\bar{y}^k \bar{x}$, where $\bar{y}$ is the class of the $2(n-1)$-dimensional generator of $\mathcal{M}((\Lambda S^n)_{S^1})$.

From 7 it follows that $H^*(\Pi S^n - \Pi^0 S^n) = 0$ for $*$ even. Let ζ be the generator of $H^{n-1}(\Pi S^n, \Pi^0 S^n)$, so $\zeta \cup w_2 \neq 0$; by 12 there exists $\eta \in H^n(BS^1 \times S^n)$ such that $\delta(\eta) = \zeta \cup w_2$. But $\eta \cup w_2^S$ does not belong to the image of j^* (again looking at 12), so $\delta(\eta \cup w_2^S) = \delta(\eta)\, w_2^S = \zeta \cup w_2^{S+1} \neq 0$, $\forall S$, and the theorem follows.

We notice that the rational Poincaré polynomial of $(\Pi S^r, \Pi^0 S^r)$ was already computed by Švarc (see [13]):

$$
P(\Pi S^r, \Pi^0 S^r)(t) = \begin{cases} \dfrac{t^{r+1}}{1-t^2} + \dfrac{t^{r-1}}{1-t^{2(r-1)}}, & \text{if } r \text{ is even} \\[2ex] \dfrac{t^{r-1}}{1-t^{r-1}} + \dfrac{t^{r+1}}{1-t^2}, & \text{if } r \text{ is odd.} \end{cases}
$$

From this, the multiplicative structures of $H^*(\Pi S^r, \Pi^0 S^r)$ and $H^*(\Pi S^r - \Pi^0 S^r)$ can be recovered by a purely dimensional argument (really we recover 13 without using Sullivan's formula, the generators of the minimal model being obvious).

Remark. We notice explicitly that $H^*(\Pi S^{2k}, \Pi^0 S^{2k})$ cannot contain a polynomial element, since it is always zero in even dimensions.

We now give an application to closed extremals of certain variational problems. Let M be a closed simply-connected manifold with the rational cohomology of a symmetric space of rank one. Let $L : TM \to \mathbf{R}$ be an " almost-Finsler variational problem " (see [7]). We call a closed extremal $c : S^1 \to M$ of L a winding extremal if there exists a sequence of natural numbers $k_n \to \infty$ such that the curves $c_n : S^1 \to M$ given by $c_n(t) = c(k_n t)$ are extremals of L.

15. THEOREM. *Under above condition,* L *admits infinitely many closed extremals or at least one winding one.*

Proof. This is an immediate consequence of 14 and the immediate generalization of Theorem of 5.2 of [7] to the almost-Finsler case.

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