Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On the boundary layer function in M. H. D. flows over a flat plate

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **63** (1977), n.3-4, p. 253–257.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1977_8_63_3-4_253_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1977.

Magnetofluidodinamica. — *On the boundary layer function in M.H.D. flows over a flat plate* (*). Nota I (**) di LUCIANO DE SOCIO e PASQUALE RENNO, presentata dal Socio G. RIGHINI.

RIASSUNTO. — Con riferimento ad un problema di perturbazione singolare che si pone per una classe di moti magnetoidrodinamici viscosi, si determina la funzione di strato limite e si perviene ad una rappresentazione asintotica uniformemente valida in ogni punto del campo d'integrazione.

INTRODUCTION

The boundary layer (B.L.) concept and related techniques are of foremost importance in several branches of mathematical physics and engineering. Even though B.L. procedures have been used for many years now, only in comparatively recent times an increasing effort has been devoted to a rigorous approach to the development of the B.L. theory. This note deals with a classic problem of magneto-hydrodynamics (M.H.D.), a field where B.L. methods have very often been applied in a somehow mechanically heuristic way. The unsteady onedimensional M.H.D. flow over a flat plate will be considered. The goal is an exact formulation of the B.L. function in the case of a small magnetic Prandtl number. Last one has been the parameter on which to rely in dealing with otherwise almost untractable situations, by means of the singular perturbation method [1, 2].

1. Let an indefinite flat plate "$\mathscr{L}$" correspond to the plane $x' y'$ of a right-handed system of Cartesian coordinates $T_0 = Ox' y' z'$, whose unit vectors are $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. A homogeneous, incompressible, electrically conducting fluid "$\mathscr{F}_\varepsilon$" of density ρ and kinematic viscosity ν is in the semispace "$\mathscr{S}$" defined by $z' > 0$. The magnetic permeability and the magnetic diffusivity of $\mathscr{F}_\varepsilon$ are μ and η respectively, and the magnetic Prandtl number is $\varepsilon^2 = \nu/\eta$. If t is the time and $\mathbf{V}$ and $\mathbf{H}$ represent the velocity vector and the magnetic field, the following dimensionless quantities can be defined: $\tau = t\,(\mu\mathscr{H}^2/\rho\nu)$; $z = z'\,(\nu/\mathscr{H})\,(\rho/\mu)$; $\boldsymbol{v} = \mathbf{V}\,(\rho/\mu)^{\frac{1}{2}}/\mathscr{H}$; $h = \mathbf{H}/\mathscr{H}$, where $\mathscr{H}$ is the modulus of a reference magnetic field. Let $\mathscr{M}_\varepsilon$ be the onedimensional M.H.D motion of $\mathscr{F}_\varepsilon$ in $\mathscr{S}$ subject to the conditions:

i) at $\tau = 0$, $\mathscr{F}_\varepsilon$ is at rest in the presence of a uniform magnetic field $\widehat{\mathscr{H}} = \mathscr{H}\boldsymbol{e}_3$,

ii) at each point of $\mathscr{L}$ and for any τ, $\boldsymbol{v}$ and $\boldsymbol{h}$ assume the assigned values $\boldsymbol{v}^*(\tau)$ and $\boldsymbol{h}^*(\tau)$ respectively, while they become infinitesimal as $z \to \infty$.

(*) Lavoro eseguito nell'ambito del G.N.F.M.-C.N.R.

(**) Pervenuta all'Accademia il 14 ottobre 1977.

Therefore $\mathscr{M}_\varepsilon$ is determined by the solution $(\boldsymbol{v}_\varepsilon, \boldsymbol{h}_\varepsilon)$ of the following problem "$\mathscr{A}_\varepsilon$" [6, 7]

$$\text{(1.1)} \qquad \begin{cases} \partial \boldsymbol{h}_\varepsilon/\partial\tau = \partial^2 \boldsymbol{h}_\varepsilon/\partial z^2 + \partial \boldsymbol{v}_\varepsilon/\partial z \\ \partial \boldsymbol{v}_\varepsilon/\partial\tau = \varepsilon^2 \partial^2 \boldsymbol{v}_\varepsilon/\partial z^2 + \partial \boldsymbol{h}_\varepsilon/\partial z \end{cases} \qquad \forall z > 0 , \forall \tau > 0$$

$$\text{(1.2)} \qquad \begin{cases} \boldsymbol{h}_\varepsilon(z, 0) = 0 \quad ; \quad \boldsymbol{v}_\varepsilon(z, 0) = 0 & \forall z > 0 \\ \boldsymbol{h}_\varepsilon(0, \tau) = \boldsymbol{h}^*(\tau) \quad ; \quad \boldsymbol{v}_\varepsilon(0, \tau) = \boldsymbol{v}^*(\tau) & \forall \tau > 0 \\ \lim_{z\to\infty} |\boldsymbol{h}_\varepsilon(z, \tau)| = 0 \quad ; \quad \lim_{z\to\infty} |\boldsymbol{v}_\varepsilon(z, \tau)| = 0 & \forall \tau > 0 \end{cases}$$

where $\boldsymbol{h}_\varepsilon \cdot \boldsymbol{e}_3 = 1$ and $\boldsymbol{v}_\varepsilon \cdot \boldsymbol{e}_3 = 0$.

Let now "$\mathscr{F}_0$" be a perfect fluid for which $\nu = 0$, whereas all the remaining properties are equal to those of $\mathscr{F}_\varepsilon$. Let "$\mathscr{M}_0$" be the onedimensional M.H.D. motion of $\mathscr{F}_0$ in $\mathscr{S}$ subject to the conditions (i) of $\mathscr{M}_\varepsilon$ and to the following boundary conditions:

i′ i′) at each point of $\mathscr{L}$ and for any τ, $\boldsymbol{h}$ assumes the assigned value $\boldsymbol{h}^*(\tau)$ and it becomes infinitesimal as $z \to \infty$.

$\mathscr{M}_0$ is then determined by the solution $(\boldsymbol{v}_0, \boldsymbol{h}_0)$ of the following problem "$\mathscr{A}_0$"

$$\text{(1.3)} \qquad \begin{cases} \partial \boldsymbol{h}_0/\partial\tau = \partial^2 \boldsymbol{h}_0/\partial z^2 + \partial \boldsymbol{v}_0/\partial z \\ \partial \boldsymbol{v}_0/\partial\tau = \partial \boldsymbol{h}_0/\partial z \end{cases} \qquad \forall z > 0 , \forall \tau > 0$$

$$\text{(1.4)} \qquad \begin{cases} \boldsymbol{h}_0(z, 0) = 0 \quad ; \quad \boldsymbol{v}_0(z, 0) = 0 & \forall z > 0 \\ \boldsymbol{h}_0(0, \tau) = \boldsymbol{h}^*(\tau) \quad ; \quad \lim_{z\to\infty} |\boldsymbol{h}_0(z, \tau)| = 0 & \forall \tau > 0 \end{cases}$$

where $\boldsymbol{h}_0 \cdot \boldsymbol{e}_3 = 1$ and $\boldsymbol{v}_0 \cdot \boldsymbol{e}_3 = 0$.

DEFINITION 1.1. "Let Q be the set $\mathscr{S} \times [0, T]$, with $T > 0$ arbitrary but bounded. A motion $(\boldsymbol{v}, \boldsymbol{h})$ is *regular* provided that $\boldsymbol{v}$ and $\boldsymbol{h}$ are continuous in Q while they are of class $C^{(1)}$ in $Q - \partial Q$ and there their second derivatives with respect to z are also continuous. A *solution* to $\mathscr{A}_\varepsilon$ [or $\mathscr{A}_0$] is any regular motion which identically satisfies (1.1)-(1.2) [or (1.3)-(1.4)]".

It's well known [10] that there exists only one $\mathscr{M}_\varepsilon$ which is regular in the sense of *Definition* 1.1.

In this note the following assumptions will be made:

j) $\boldsymbol{v}^*(\tau)$ and $\boldsymbol{h}^*(\tau)$ are generally continuous in $[0, \infty[$ and summable in any bounded interval,

jj) $\boldsymbol{v}^*(\tau)$ and $\boldsymbol{h}^*(\tau)$ are bounded in $]0, T]$.

In [7] P. Renno has shown that, $\forall \varepsilon > 0$, there exists only one solution of $\mathscr{A}_\varepsilon$ and this solution was explicitly evaluated. On the other hand, several formulations of the solution to $\mathcal{A}_0$ are known (see, for instance, [3] and [5]). By comparison of $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ some interesting questions arise which will be investigated, namely 1st) if $\mathcal{M}_\varepsilon \to \mathcal{M}_0$ as $\varepsilon \to 0$, 2 nd) the error in approximating $\mathcal{M}_\varepsilon$ by means of $\mathcal{M}_0$. One should note that this last approximation can not be uniformely valid $\forall z$, due to the lack of an adherence condition for $\mathcal{F}_0$ on $\mathcal{L}$. The second question is a classic singular perturbation problem which can be solved only after a suitable boundary layer function is found which approximates $\mathcal{M}_\varepsilon$ also in the layer adjacent to $\mathcal{L}$.

The main result of this paper is the theorem that follows which will be demonstrated after determining suitable majorant functions of the solution $(\boldsymbol{v}_\varepsilon , \boldsymbol{h}_\varepsilon)$ to $\mathcal{A}_\varepsilon$ and after the demonstration of a lemma.

THEOREM 1.1. *In each point of* Q *the relations*

$$(1.5) \qquad \boldsymbol{v}_\varepsilon = \boldsymbol{v}_0 + \boldsymbol{b} + \varepsilon \boldsymbol{r} \quad , \quad \boldsymbol{h}_\varepsilon = \boldsymbol{h}_0 + \varepsilon \boldsymbol{s}$$

hold, where $(\boldsymbol{v}_0 , \boldsymbol{h}_0)$ *is the solution to problem* $\mathcal{A}_0$, $\boldsymbol{r}$ *and* $\boldsymbol{s}$ *are bounded functions,* $\forall \varepsilon \geq 0$, *vanishing at* $z = 0$, *whereas* $\boldsymbol{b}$ *is a boundary layer function such that*

$$(1.6) \qquad \lim_{z \to 0} [\boldsymbol{v}_0 (z , \tau) + \boldsymbol{b} (z , \tau ; \varepsilon)] = \boldsymbol{v}^* (\tau) , \qquad \forall \varepsilon > 0 .$$

Furthermore, if $\boldsymbol{v}^* (\tau)$ *and* $\boldsymbol{h}^* (\tau)$ *are bounded for* $\tau \to \infty$ *then,* $\forall \tau \geq 0$, *the following bound to* $\boldsymbol{r}$ *and* $\boldsymbol{s}$ *holds*

$$(1.7) \qquad | \boldsymbol{r} (z , \tau ; \varepsilon) | + | \boldsymbol{s} (z , \tau ; \varepsilon) | < \sigma (z , \varepsilon) , \qquad \forall \tau > 0$$

where $\sigma (z , \varepsilon)$ *is bounded* $\forall \varepsilon \geq 0$ *and such that* $\sigma (0 , \varepsilon) = 0$".

The paper ends with an application where the case $\boldsymbol{v}^*$ and $\boldsymbol{h}^*$ both constants is considered, and with a discussion of the B.L. thickness.

Comment 1.1. Nardini [4] shows that, as $\eta \to 0$, $\mathcal{M}_0$ tends to a wavy motion propagating with the Alfven speed $(\mu/\rho)^{\frac{1}{2}} \mathrm{H}_0$. The more regular are the boundary values, the better is the approximation of $\mathcal{M}_0$ by means of the mentioned wave propagation as $\eta \to 0$. This result, together with *Theorem 1.1*, enables one to rigorously approximate $\mathcal{M}_\varepsilon$ in all the region of the so called viscous-magnetic B.L.

2. The solution to $\mathcal{A}_\varepsilon$ as given in [7] can be re-arranged in the form

$$(2.1) \qquad \boldsymbol{v}_\varepsilon = \mathrm{F}_2 * \boldsymbol{v}^* - \mathrm{F}_1 * \boldsymbol{h}^* \quad , \quad \boldsymbol{h}_\varepsilon = \mathrm{F}_3 * \boldsymbol{h}^* - \varepsilon \mathrm{F}_1 * \boldsymbol{v}^*$$

where the symbol $*$ stays for the usual convolution integral and the functions $\mathrm{F}_i (z , \tau ; \varepsilon)$, $(i = 1 , 2 , 3)$, are defined by

$$(2.2) \qquad \mathrm{F}_1 (z , \tau ; \varepsilon) = \frac{z^2 e^{-\delta t} (\pi\tau)^{-\frac{1}{2}}}{4 \varepsilon (1 + \varepsilon) \tau} \int_0^2 (\alpha u + \gamma) \, e^{-z^2(\alpha u + \gamma)^2/4\tau} \, \mathrm{I}_0 \{\xi z [u (2 - u)]^{\frac{1}{2}}\} \, du$$

$$(2.3)\qquad F_2(z,\tau;\varepsilon) = \frac{(2\alpha+\gamma)\, z e^{-\delta\tau}(\pi\tau)^{-\frac{1}{2}}}{2\tau} e^{-(2\alpha+\gamma)^2 z^2/4\tau} + \frac{\xi z^2 e^{-\delta\tau}(\pi\tau)^{-\frac{1}{2}}}{4\tau}\cdot$$

$$\cdot \int_0^2 (\alpha u+\gamma)\, e^{-z^2(\alpha u+\gamma)^2/4\tau}\,[u/(2-u)]^{\frac{1}{2}}\, I_1\{\xi z\,[u(2-u)]^{\frac{1}{2}}\}\,du$$

$$(2.4)\quad F_3(z,\tau;\varepsilon) = \frac{\gamma z e^{-\delta\tau}(\pi\tau)^{-\frac{1}{2}}}{2\tau} e^{-\gamma^2 z^2/4\tau} + \frac{\xi z^2 e^{-\delta\tau}(\pi\tau)^{-\frac{1}{2}}}{4\tau}\int_0^2 (\alpha u+\gamma)\, e^{-z^2(\alpha u+\gamma)^2/4\tau}\cdot$$

$$\cdot [(2-u)/u]^{\frac{1}{2}}\, I_1\{\xi z\,[u(2-u)]^{\frac{1}{2}}\}\,du\,.$$

Here I_i, $(i = 0, 1, 2, \cdots)$ are the modified Bessel function of the first kind and of order i and the constants are: $\alpha = |1-\varepsilon|/2\varepsilon$; $\beta = (1+\varepsilon)/2\varepsilon$; $\gamma = \beta - \alpha$; $\delta = (1+\varepsilon)^{-2}$; $\xi = (\delta/\varepsilon)^{\frac{1}{2}}$.

As far as problem $\mathscr{A}_0$ is concerned, its solution has been given different formulations. Here the solution of $\mathscr{A}_0$ will be expressed by means of definite integrals. In particular, let

$$(2.5)\qquad G_1(z,\tau) = \frac{z^2 e^{-\tau}(\pi\tau)^{-\frac{1}{2}}}{2\tau}\int_1^\infty e^{-z^2u^2/4\tau}\, u I_0\{2z(u-1)^{\frac{1}{2}}\}\,du$$

$$(2.6)\qquad G_3(z,\tau) = \frac{z e^{-\tau-z^2/4\tau}(\pi\tau)^{-\frac{1}{2}}}{2\tau} + \frac{z^2 e^{-\tau}(\pi\tau)^{-\frac{1}{2}}}{2\tau}\int_1^\infty e^{-z^2u^2/4\tau}\cdot$$

$$\cdot\, u(u-1)^{-\frac{1}{2}}\, I_1\{2z(u-1)^{\frac{1}{2}}\}\,du\,;$$

then, under the assumptions (j) and (jj) for $h^*(\tau)$ the unique solution to $\mathscr{A}_0$ is

$$(2.7)\qquad v_0(z,\tau) = -G_1(z,\tau) * h^*(\tau)\quad ;\quad h_0 = G_3(z,\tau) * h^*(\tau)\,.$$

Comment 2.1. Observe that from (2.7) the following equation can be obtained

$$(2.8)\qquad \lim_{z\to 0} v_0(z,\tau) = -e^{-\tau}(\pi\tau)^{-\frac{1}{2}} * h^*(\tau)\,.$$

Some useful relations will now be given. If y is a positive real variable, then [8]

$$(2.9)\qquad I_n(2y) \le y^n I_0(2y) \le y^n e^{2y}\,.$$

Furthermore, let p and q be two parameters and $u = (p^2+q^2)^{\frac{1}{2}}$. If one considers the principal determination of the radicals, the following relations can be obtained from the properties of Bessel functions [9] with a little manipulation

$$(2.10) \qquad \int_0^2 e^{-pv}\, I_0\{q\,[v\,(2-v)]^{\frac{1}{2}}\}\, dv = (2/u)\, e^{-p} \sinh u$$

$$(2.11) \qquad \int_0^2 e^{-pv}\, [v/(2-v)]^{\frac{1}{2}}\, I_1\{q\,[v\,(2-v)]^{\frac{1}{2}}\}\, dv \leq (2/q)\,(1-p/u)\, e^{-p} \sinh u$$

$$(2.12) \qquad \int_0^2 e^{-pv}\, v I_0\{q\,[v\,(2-v)]^{\frac{1}{2}}\}\, dv \leq (2\, e^{-p}/u^2)\, [(p \sinh u)/u + e^u\,(u-p)/2]$$

$$(2.13) \qquad \int_0^2 e^{-pv}\, v\, [v/(2-v)]^{\frac{1}{2}}\, I_1\{q\,[v\,(2-v)]^{\frac{1}{2}}\}\, dv \leq$$

$$\leq (2\, e^{-p+u}/q)\,(q^2/2\, u^3 - p/u - q^2/2\, u^2 + 1)\,.$$

REFERENCES

[1] G. A. LYUBIMOV (1965) – *The Magnetohydrodynamic Boundary Layer*, « Magnit. Gidrod. », *1*, 3–30.

[2] L. DE SOCIO (1974) – *Viscous Magnetic Boundary Layers*, « Z.A.M.M. », *54*, 373–381.

[3] R. NARDINI (1953) – *Soluzione di un problema al contorno della magnetoidrodinamica*, « Ann. Mat. Pura e Appl. », (4), *35*, 269–290.

[4] R. NARDINI (1954) – *Su comportamento asintotico della soluzione di un problema al contorno della magnetofluidodinamica*, « Rend. Acc. Naz. Lincei », ser. VII, *16*.

[5] G. S. S. LUDFORD (1959) – *Rayleigh's Problem in Hydromagnetics: the Impulsive Motion of a Pole-piece*, « Arch. Rat. Mech. Ann. », *3*, 14–27.

[6] J. ROSCISZEWSKI (1964) – *Magnetohydrodynamic Rayleigh Problem for Magnetic Prandtl Number close to One*, « Arch. Rat. Mech. An. », *16*, 230–237.

[7] P. RENNO (1974) – *Sulla stabilità asintotica in presenza di perturbazioni al contorno per una classe di moti magnetoidrodinamici*, « Rend. Acc. Sc. fis. mat. della Soc. Naz. Sc. Lett., Napoli », ser. IV, *41*, 435–468.

[8] G. N. WATSON (1966) – *A treatise on the Theory of Bessel Functions*, Cambridge, at the Univ. Press.

[9] A. ERDELYI, W. MAGNUS, F. OBERHETTINGER and F. G. TRICOMI (1954) – *Tables of Integral Transforms*, I, McGraw-Hill.

[10] R. NARDINI (1970) – *Su un caso particolare di stabilità in media della magnetoidrodinamica*, « Ann. Mat. Pura e Appl. », (4), *84*, 95–100.