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Andrzej Borzymowski

The uniqueness of the solution of a tangential derivative problem for a system of non-linear parabolic equations

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Equazioni a derivate parziali. — *The uniqueness of the solution of a tangential derivative problem for a system of non-linear parabolic equations.* Nota (*) di ANDRZEJ BORZYMOWSKI, presentata dal Corrisp. G. CIMMINO.

RIASSUNTO. — La Nota tratta l'unicità della soluzione del primo problema di Fourier per un sistema di equazioni paraboliche del secondo ordine non lineari con una condizione al contorno contenente le derivate delle funzioni incognite in direzioni tangenti alla frontiera del dominio.

1. The uniqueness of solutions of Fourier problems for second-order parabolic equations and systems of such equations has been examined by many Authors (see [1]-[11]) under the assumption that the boundary conditions of the problems do not contain the derivatives of the unknown functions in the directions not entering the closure of the considered domain. To the best of our knowledge the case of other directions has not been considered so far.

In this paper we prove the uniqueness of the solution of the first Fourier problem for a system of non-linear second-order parabolic equations with a non-linear boundary condition containing the derivatives of the unknown functions in arbitrary directions tangent to the frontier of the domain. Let us note that in the previous papers concerning the uniqueness of the solution of the first Fourier problem for systems of parabolic equations (see [1] and [9]-[11]) only linear boundary conditions were dealt with.

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2. Let R^{m+1} be the space-time of the points $(x, t) = (x_1, x_2, \cdots, x_m, t)$, where $m \geq 2$, and let D_h denote a $(m+1)$-dimensional non-cylindrical domain whose boundary consists of two m-dimensional closed domains $\overline{\Omega}_0$ and $\overline{\Omega}_h$ placed in the hyperplanes $t = 0$ and $t = h$ respectively, where h is a finite positive number, and a m-dimensional lateral surface σ_h of equation $\Gamma(x, t) = 0$ situated between these hyperplanes. We shall use the notation $\Sigma = \sigma_h \cup \overline{\Omega}_0$.

Consider the following system of partial differential equations

$$\frac{\partial u_r}{\partial t} = F_r\left(x, t, u_i, \frac{\partial u_r}{\partial x_j}, \frac{\partial^2 u_r}{\partial x_j \partial x_k}\right) \tag{1}$$

$$\left(r = 1, 2, \cdots, n; u_i = u_1, \cdots, u_n;\right.$$

$$\left.\frac{\partial u_r}{\partial x_j} = \frac{\partial u_r}{\partial x_1}, \cdots, \frac{\partial u_r}{\partial x_m}; \quad \frac{\partial^2 u_r}{\partial x_j \partial x_k} = \frac{\partial^2 u_r}{\partial x_1^2}, \frac{\partial^2 u_r}{\partial x_1 \partial x_2}, \cdots, \frac{\partial^2 u_r}{\partial x_m^2}\right),$$

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where $F_r(x, t, y_i, z_j, z_{jk})$ are given functions defined for $(x, t) \in D_h$ and arbitrary y_i, z_j, z_{jk}. We shall base on the following definition due to J. Szarski (see [10], p. 133):

A solution $u_i(x, t)$ of system (1) is said to be parabolic in D_h if for each system of numbers $z_{jk}, \bar{z}_{jk}$ $(j, k = 1, \cdots, m)$, $z_{jk} = z_{kj}$; $\bar{z}_{jk} = \bar{z}_{kj}$, such that the form

$$\sum_{j,k=1}^{m} (z_{jk} - \bar{z}_{jk}) \lambda_j \lambda_k$$

is non-positive for all vectors $(\lambda_1, \cdots, \lambda_m)$, the inequality

$$F_r\left(x, t, u_i(x, t), \frac{\partial u_r}{\partial x_j}(x, t), z_{jk}\right) - \tag{2}$$

$$- F_r\left(x, t, u_i(x, t), \frac{\partial u_r}{\partial x_j}(x, t), \bar{z}_{jk}\right) \leq 0$$

holds true for $(x, t) \in D_h$; $r = 1, 2, \cdots, n$.

Let us assume that to each point $(x, t) \in \sigma_h$ there correspond unitary vectors $s_1, s_2, \cdots, s_q$ $(1 \leq q \leq m - 1)$ tangent to σ_h and parallel to the plane $t = 0$.

The problem to be examined, from now on called the (F_1)-problem, consists in finding a parabolic solution $u_i(x, t)$ of system (1) that is regular in D_h(1), possesses the tangential derivatives $\frac{du_i}{ds_j}(x, t)$ $(j = 1, 2, \cdots, q)$ at points of σ_h and satisfies the initial condition

$$u_r(x, 0) = \mathscr{C}_r(x) \tag{3}$$

for $x \in \bar{\Omega}_0$ and the boundary condition

$$u_r(x, t) = G_r\left[x, t, u_i(x, t), \frac{du_r}{ds_j}(x, t)\right] \tag{4}$$

for $(x, t) \in \sigma_h$ $(r = 1, 2, \cdots, n; j = 1, 2, \cdots, q)$, where $\mathscr{C}_r(x)$ and $G_r(x, t, y_i, w_j)$ are given functions defined on the sets $x \in \bar{\Omega}_0$ and $\{(x, t) \in \sigma_h;$ $y_i \in (-\infty, +\infty), w_j \in (-\infty, +\infty)\}$ respectively.

The Theorem below is valid.

THEOREM 1. *We assume the following.*

1) *The function* $\Gamma(x, t)$ *is defined and of class* C^1 *in a closed domain*

(1) I.e. continuous on the set $D_h \cup \Sigma$ and possessing continuous derivatives $\frac{\partial u_i}{\partial x_j}$, $\frac{\partial^2 u_i}{\partial x_j \partial x_k}$ and $\frac{\partial u}{\partial t}$ in D_h.

D_h^* *containing the closure* $\bar{D}_h$ *of* D_h, *satisfies the condition* $\operatorname{grad}_x^2 \Gamma(x, t) = \sum_{i=1}^{m} \left[\frac{\partial \Gamma}{\partial x_i}(x, t) \right]^2 > 0$ *for* $(x, t) \in \sigma_h$ *and possesses bounded and continuous derivatives* $\frac{\partial^2 \Gamma}{\partial x_j \partial x_k}(x, t)$ $(j, k = 1, 2, \cdots, m)$ *in* D_h.

2) *The functions* $F_r(x, t, y_i, z_j, z_{jk})$ *and* $G_r(x, t, y_i, w_j)$ *fulfill the conditions* (2)

$$F_r(x, t, y_i, z_j, z_{jk}) - F_r(x, t, \bar{y}_i, \bar{z}_j, \bar{z}_{jk}) \leq \leq L_0 \sum_{i=1}^{m} |y_i - \bar{y}_j| + L_1 \sum_{j=1}^{m} |z_j - \bar{z}_j| + L_2 \sum_{j,k=1}^{m} |z_{jk} - \bar{z}_{jk}| ; \tag{5}$$

$$G_r(x, t, y_i, w_j) - G_r(x, t, \bar{y}_j, w_j) \leq L_3 \sum_{i=1}^{n} |y_i - \bar{y}_i| \tag{6}$$

$(r = 1, 2, \cdots, n; y_r \geq \bar{y}_r)$, *where* L_0, L_1, L_2, L_3 *are positive constants, and the inequality*

$$nL_3 < 1 \tag{7}$$

holds true.

Then the problem (F_1) possesses at most one solution of class C^1 on $D_h \cup \sigma_h$.

In order to prove Theorem 1 we shall need the following Lemma:

LEMMA 1. *Let* Ω *be a m-dimensional domain in* R^m*-space* $(m \geq 2)$ *and let a* $(m - 1)$*-dimensional set* S *constitute the boundary or a part of the boundary of* Ω. *Assume that*

1) S *is a regular closed surface or a regular open surface patch of class* C^1;

2) $u(x)$ *is a function of class* C^1 *on the set* $\Omega \cup S$, *the first-order partial derivatives of* $u(x)$ *at the points of* S *being defined by*

$$\frac{\partial u}{\partial x_i} \stackrel{\text{def}}{=} \lim_{\bar{x} \to x} \frac{\partial u}{\partial x_i}(\bar{x}) \qquad (i = 1, 2, \cdots, m; x \in S; \bar{x} \in \Omega);$$

3) $u(x)$ *attains its upper or lower bound at a point* $x_0 \in S$.

Under these assumptions, for each vector s *tangent to* S *at* x_0 *the equality*

$$\frac{du}{ds}(x_0) = 0 \tag{8}$$

holds.

(2) Conditions (5) and (6) are also called the $(\mathscr{L})$-conditions (see [1], 249). Note that they somewhat weaker than the Lipschitz condition.

Proof of Lemma 1. Note first of all that by assumptions 1) and 2), there exist a ball K of centre x_0 and radius $r > 0$ and a function $u^*(x)$ such that $u^*(x)$ is of class C^1 on K and $u^*(x) = u(x)$ for $x \in (\Omega \cup S) \cap K$. Furthermore, les us observe that the following equality

$$\frac{du^*}{ds}(x_0) = \lim_{l \to 0} \frac{u^*(x) - u^*(x_0)}{l}$$

is satisfied, where $l = (\widehat{x_0 x})$ is the relative value of the arc $\widehat{x_0 x}$ of a directed curve L obtained in the intersection of a sufficiently small portion of the surface patch $S \cap K$ and the two-dimensional plane Π passing through the vector s and the normal to S at x_0 (we assume that s and L are of the same direction). Hence, we can assert that

$$\frac{du}{ds}(x_0) = \lim_{l \to 0} \frac{u(x) - u(x_0)}{l} \tag{9}$$

holds true. The required relation (8) is an immediate consequence of (9).

Proof of Theorem 1. Following papers [1], [2] we suppose there exist two solutions, $u_i^{(1)}$ and $u_i^{(2)}$ $(u_i^{(k)} = u_1^{(k)}, \cdots, u_n^{(k)}; k = 1, 2)$, and we set

$$u_i^{(k)}(x, t) = v_i^{(k)}(x, t) \cdot H(x, t) \tag{10}$$

where

$$H(x, t) = \exp\left[\frac{\Gamma^2(x, t)}{1 - \mu t} + \nu t\right], \tag{11}$$

μ and ν being positive constants to be determined later. On denoting $u_i = u_i^{(1)} - u_i^{(2)}$; $v_i = v_i^{(1)} - v_i^{(2)}$, we have from (10), (11) $u_i = v_i \cdot H(x, t)$.

Now, let τ be a number arbitrarily fixed in the interval $(0, h)$ and introduce the notation $D_\tau = D_h \cap (0 < t < \tau)$; $\tilde{D}_\tau = D_h \cap (0 < t \leq \tau)$; $\tilde{\sigma}_\tau = \sigma_h \cap \cap (0 < t \leq \tau)$. Set $M = \max_{(i)} \sup_{\bar{D}_\tau} |v_i(x, t)|$, where $\bar{D}_\tau$ is the closure of D_τ. Evidently, there exist an index i_* and a point $(x_*, t_*) \in \bar{\Gamma}_\tau$ such that $M = |v_{i_*}(x_*, t_*)|$. It can be shown in a way analogous to that in [1], 251–253 that if $h < h_0$ (h_0 being a sufficiently small positive number) and the parameters μ and ν are chosen as follows

$$\mu = 4 A^2 L_2 + 1; \tag{12}$$

$$\nu = \frac{1}{\gamma^2} [1 + (C + BL_2 + AL_1)^2 + 2 A^2 L_2 + nL_0], \tag{13}$$

where $\sum_{j=1}^{m} \left| \frac{\partial \Gamma}{\partial x_j} \right| \leq A$; $\sum_{j,k=1}^{m} \left| \frac{\partial^2 \Gamma}{\partial x_j \partial x_k} \right| \leq B$; $\left| \frac{\partial \Gamma}{\partial t} \right| \leq C$; $\gamma = 1 - \mu h$, then the relation $(x_*, t_*) \in \bar{\Omega}_0 \cup \tilde{D}_\tau$ implies $M = 0$. We shall prove that $M = 0$ also in case when $(x_*, t_*) \in \tilde{\sigma}_\tau$, $(h < h_0)$.

By Lemma 1 above we have

$$\frac{d}{ds_j} v_{i_*}(x_*, t_*) = 0$$

whence

$$\frac{d}{ds_j} v_{i_*}^{(1)}(x_*, t_*) = \frac{d}{ds_j} v_{i_*}^{(2)}(x_*, t_*) \tag{14}$$

$(j = 1, 2, \cdots, q)$. Also, the following relations

$$\frac{d}{ds_j} H(x_*, t_*) = \frac{2\varepsilon H(x_*, t_*)}{1 - \mu t_*} \Gamma(x_*, t_*) \left| \operatorname{grad}_x \Gamma(x, t) \right| \cos(\bar{n}, s_j) = 0 \tag{15}$$

$(\varepsilon = \pm 1)$ are valid. In virtue of (14) and (15) we can assert that

$$\frac{d}{ds_j} [v_{i_*}^{(1)}(x_*, t_*) \cdot H(x_*, t_*)] = \frac{d}{ds_j} [v_{i_*}^{(2)}(x_*, t_*) \cdot H(x_*, t_*)] \tag{16}$$

holds true.

Using (4) with $r = i_*$ and basing on (6), (10) and (16) we get the inequality

$$| v_{i_*}(x_*, t_*) | \, H(x_*, t_*) \leq n L_3 \, | v_{i_*}(x_*, t_*) | \cdot H(x_*, t_*)$$

and as a consequence (see (7)) we obtain $|v_i(x, t)| \leq 0$, as required.

If $h \geq h_0$, we set $\tilde{t} = t + jh_0$, (where $j = 1, 2, \cdots$) and hence prove our theorem successively for the parts of D_h contained in the zones $jh_0 \leq t \leq \leq (j + 1) h_0$.

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