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Integral stability of differential systems and perturbation of Lyapunov functions

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Equazioni differenziali ordinarie. — *Integral stability of differential systems and perturbation of Lyapunov functions*. Nota (*) di OLUSOLA AKINYELE, presentata dal Socio G. SANSONE.

RIASSUNTO. — L'Autore con l'impiego di funzioni di Lyapunov dà criteri sufficienti abbastanza generali per ottenere proprietà di stabilità integrale non uniforme delle soluzioni di un sistema differenziale.

§ 1. INTRODUCTION

Results of non-uniform properties of differential systems under very weak hypotheses are not easy to obtain and one useful and important tool recently introduced in [2] is to perturb the Lyapunov functions involved. The importance of this technique in the study of non-uniform properties of differential systems was demonstrated in connection with equiboundedness and equistability results under considerably relaxed assumptions. This method was also utilized in [1] to give a treatment of perfect equistability, perfect equi-asymptotic stability and strong equistability of solutions of differential systems under relaxed conditions.

To obtain results on non-uniform integral stability properties of differential systems by means of Lyapunov functions, one must generally assume conditions everywhere in R^n [3]. In this paper, we investigate the non-uniform integral stability properties of differential systems under weaker assumptions, by means of Lyapunov functions and the theory of differential inequalities. We do not wish to require that our Lyapunov functions be defined everywhere in R^n, so we perturb the Lyapunov functions suitably to obtain the desired result. Our results also strengthen the belief that the technique of perturbing Lyapunov functions is destined to play an increasingly important role in the study of non-uniform properties of solutions of differential system as well as the preservation of such properties under constantly acting perturbations.

§ 2. PRELIMINARIES

Let R^n be the n-dimensional Euclidean space and E any subset of R^n. Denote by $\overline{E}$, E^c and ∂E the closure, the complement and the boundary of E respectively. For any $\rho > 0$, define $S(\rho) = \{x \in R^n : \|x\| < \rho\}$ where $\|\cdot\|$ is any convenient norm. Let $R^+ = [0, \infty)$.

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Consider a system of ordinary differential equations of the form

$$\frac{dx}{dt} = f(t, x) \qquad x(t_0) = x_0 , \tag{1}$$

where $f \in C(R^+ \times R^n, R^n)$ and $C(R^+ \times R^n, R^n)$ is the space of continuous functions with domain $R^+ \times R^n$ and range R^n; and the perturbed system

$$\frac{dx}{dt} = f(t, x) + R(t, x), x(t_0) = x_0 , \tag{2}$$

where $f, R \in C(R^+ \times R^n, R^n)$.

The following are the definitions which we need in this paper.

DEFINITION 2.1. A function ϕ is said to belong to the class $\mathcal{K}$ if $\phi \in C([0, \rho), R^+), \phi(0) = 0$ and $\phi(r)$ is strictly monotone increasing in r.

DEFINITION 2.2. The trivial solution of (1) is:

I_1: Equi-integrally stable if for every $\alpha \geq 0$ and $t_0 \in R^+$ there exists a positive function $\beta = \beta(t_0, \alpha)$ which is continuous in t_0 for each α and $\beta \in \mathcal{K}$ for each t_0 such that for every solution $x(t, t_0, x_0)$ of the perturbed system (2). $x(t, t_0, x_0) \in S(\beta)\, t \geq t_0$, holds provided that $x_0 \in \overline{S(\alpha)}$ and for every $T > 0$,

$$\int_{t_0}^{t_0+T} \sup_{x \in \overline{S(\beta)}} \| R(s, x(s)) \| \, ds \leq \alpha .$$

I_2: Uniformly-integrally stable if β in I_1 is independent of t_0.

I_3: Equi-asymptotically integrally stable if I_1 holds and for every $\varepsilon > 0$ $\alpha \geq 0$ and $t_0 \in R^+$ there exist positive numbers $T = T(t_0, \alpha, \varepsilon)$ and $\gamma = \gamma(t_0, \alpha, \varepsilon)$ such that for every solutions of the system (2) $x(t, t_0, x_0) \in S(\varepsilon)$ for $t \geq t_0 + T$ holds provided that $x_0 \in \overline{S(\alpha)}$ and

$$\int_{t_0}^{\infty} \sup_{x \in \overline{S(\beta)}} \| R(s, x(s)) \| \, ds \leq \gamma .$$

I_4: Uniformly-asymptotically integrally stable if the T and γ in I_3 are independent of t_0 and I_2 holds. Consider the scalar differential equation

$$\frac{du}{dt} = g(t, u) \qquad u(t_0) = u_0 \tag{3}$$

together with the perturbed differential equation

$$\frac{du}{dt} = g(t, u) + \phi(t) \qquad u(t_0) = u_0 \tag{4}$$

where $g \in C(R^+ \times R^+, R)$ and $\phi \in C(R^+, R^+)$.

Corresponding to the integral stability definitions for the system (1) we have similar concepts of integral stability for the system (3). We give one such definition since the remaining can be formulated accordingly.

DEFINITION 2.3. The trivial solution of (3) is said to be Equi-integrally stable if for every $\alpha_1 \geq 0$, $t_0 \in R^+$ there exists a positive function $\beta_1 = \beta_1(t_0, \alpha_1)$ which is continuous in t_0 for each α_1 and $\beta_1 \in \mathcal{K}$ for each t_0 such that which ever be the function $\phi \in C(R^+, R^+)$ with

$$\int_{t_0}^{t_0+T} \phi(s)\, ds \leq \alpha_1,$$

for every $T > 0$, every solution $u(t, t_0, u_0)$ of the perturbed scalar equation (4) satisfies the inequality $u(t, t_0, u_0) < \beta$ for $t \geq t_0$ provided that $u_0 \leq \alpha_1$.

We now give a theorem which permits us to study equi-integral stability properties of the differential system (1) under assumptions weaker than known result [3].

THEOREM 2.4. *Assume that*

(i) $E \subset R^n$ *is compact*, $V_1 \in C(R^+ \times \overline{E}^c, R^+)$, $V_1(t, x)$ *is locally Lipschitzian in* x *bounded for* $(t, x) \in R^+ \times \partial E$ *and*

$$D^+ V_1(t, x)_{(1)} = \overline{\lim_{h \to 0^+}} \frac{1}{h} [V_1(t + h, x + hf(t, x)) - V_1(t, x)] \leq$$
$$\leq g_1(t, V_1(t, x)) \qquad \text{for} \quad (t, x) \in R^+ \times \overline{E}^c$$

where $g_1 \in C(R^+ \times R^+, R)$; $g_1(t, 0) = 0$;

(ii) $V_2 \in C(R^+ \times S^c(\rho), R^+)$, $V_2(t, x)$ *is locally Lipschitzian in* x,

$$b(\|x\|) \leq V_2(t, x) \leq a(\|x\|) \qquad \text{for} \quad (t, x) \in R^+ \times S^c(\rho)$$

where $a, b \in C([\rho, \infty), R^+)$ *such that* $b(u) \to \infty$ *as* $u \to \infty$ *and for* $(t, x) \in R^+ \times S^c(\rho)$,

$$D^+ V_1(t, x)_{(1)} + D^+ V_2(t, x)_{(1)} \leq g_2(t, V_1(t, x) + V_2(t, x))$$

where $g_2 \in C(R^+ \times R^+, R)$; $g_2(t, 0) = 0$; $g_2(t, v)$ *is monotone nonincreasing in* v *for each* t;

(iii) *the trivial solution of the scalar differential equation*

$$\frac{du}{dt} = g_1(t, u), \quad u(t_0) = u_0 \geq 0 \tag{5}$$

is equi-integrally stable and every solution of the scalar differential equation

$$\frac{dv}{dt} = g_2(t, v), \quad v(t_0) = v_0 \geq 0 \tag{6}$$

is uniformly bounded.

Then the trivial solution of (1) *is equi-integrally stable.*

Proof. There exist $M_1 > 0$, and $M_2 > 0$ such that

$$(7) \qquad |V_1(t,x) - V_1(t,y)| \leq M_1 \|x-y\| \quad \text{and} \quad |V_2(t,x) - V_2(t,y)| \leq M_2 \|x-y\|.$$

Suppose $x(t, t_0, x_0)$ is any solution of (2), then by assumption (ii) and (7), we have

$$(8) \qquad D^+ V_1(t,x)_{(2)} + D^+ V_2(t,x)_{(2)} \leq g_2(t, V_1(t,x) + V_2(t,x)) + \lambda(t)$$

$$\text{for } (t,x) \in R^+ \times \overline{E^c} \cap S^c(\rho), \quad \text{where} \quad \lambda(t) = (M_1 + M_2) \|R(t,x)\|.$$

Let $\rho_0 > 0$ and define $S(E, \rho_0) = \{x \in R^n : d(x, E) < \rho_0\}$ where $d(x, E) = \inf_{y \in E} \|x - y\|$. E is compact therefore we can find $\rho > 0$ such that $S(R, \rho_0) \subset S(\rho)$.

Given $t_0 \in R^+$ and $\alpha \geq \rho$, define

$$\alpha_0 = \max \{V_1(t_0, x_0) : x_0 \in \overline{S(\alpha) \cap E^c}\} \qquad \text{and} \qquad \alpha^* \geq V_1(t, x)$$

for $(t, x) \in R^+ \times \partial E$.

Let $\alpha_1 = \alpha_1(t_0, \alpha) = \max\{\alpha_0, \alpha^*, (M_1 + M_2)\alpha\}$ and assume that the trivial solution of (5) is equi-integrally stable, then given $\alpha_1 \geq 0$, and $t_0 \in R^+$ there exists $\beta_1 = \beta_1(t_0, \alpha_1)$ continuous in t_0 for each α_1 and $\beta_1 \in \mathscr{K}$ for each t_0 such that for every solution $u(t, t_0, u_0)$ of

$$(9) \qquad \frac{du}{dt} = g_1(t, u) + \phi(t), \; u(t_0) = u_0,$$

with $g_1 \in C(R^+ \times R^+, R)$ and $\phi \in C(R^+, R^+)$,

$$(10) \qquad u(t, t_0, u_0) < \beta_1 \, t \geq t_0$$

holds whenever $u_0 \leq \alpha_1$ and for every $T > 0$

$$(11) \qquad \int_{t_0}^{t_0+T} \phi(s)\, ds \leq \alpha_1.$$

Set $\alpha_2 = \max\{\beta_1 + a(\alpha), (M_1 + M_2)\alpha\}$, then given $\alpha_2 \geq 0$ the uniform boundedness of solutions of (6) implies the existence of $\beta_2 = \beta_2(\alpha_2)$ such that

$$(12) \qquad v(t, t_0, v_0) < \beta_2(\alpha_2) \qquad \text{for} \quad t \geq t_0 \qquad \text{provided} \quad v_0 \leq \alpha_2$$

where $v(t, t_0, v_0)$ is any solution of (6). Moreover since $b(u) \to \infty$ as $u \to \infty$ there exists a $\beta = \beta(t_0, \alpha)$ such that

$$(13) \qquad b(\beta) > \beta_2(\alpha_2) + \alpha_2.$$

With β, every solution $x(t, t_0, x_0)$ of (2) satisfies $x(t, t_0, x_0) \in S(\beta)$ for $t \geq t_0$ provided $x_0 \in \overline{S(\alpha)}$ and for every $T > 0$,

$$\int_{t_0}^{t_0+T} \sup_{x \in \overline{S(\beta)}} \| R(s, x(s)) \| ds \leq \alpha .$$

Suppose not; then there exists a solution $x(t, t_0, x_0)$ of (2) such that for some $t_1 > t_0$,

$$\| x(t_1, t_0, x_0) \| = \beta , \tag{14}$$

and either

$$x(t, t_0, x_0) \in E^c \qquad \text{for} \quad t \in [t_0, t_1] \tag{15}$$

or

$$\begin{cases} \text{there exists } t^* \geq t_0 & \text{such that } x(t^*, t_0, x_0) \in \partial E \\ \text{and } x(t, t_0, x_0) \in E^c & \text{for } t \in [t^*, t_1] \text{ since } S(E, \rho) \subset S(\alpha). \end{cases} \tag{16}$$

If (15) holds, then there exists $t_2 > t_0$ such that

$$\begin{cases} x(t_2, t_0, x_0) \in \partial S(\alpha),\ x(t_1, t_0, x_0) \in \partial S(\beta) & \text{and} \\ x(t, t_0, x_0) \in S^c(\alpha) & \text{for } t \in [t_2, t_1]. \end{cases} \tag{17}$$

Set $u_0 = V_1(t_0, x_0)$ and $v_0 = V_1(t_2, x(t_2)) + V_2(t_2, x(t_2))$ and define

$$m(t) = V_1(t, x)(t, t_0, x_0)) + V_2(t, x(t, t_0, x_0)) -$$
$$-(M_1 + M_2) \int_{t_0}^{t} \| R(s, x(s)) \| ds$$

for $t \in [t_2, t_1]$. Then by (8) and the monotone non-increasing property of g_2 we obtain, for $t \in [t_2, t_1]$

$$D^+ m(t) \leq D^+ V_1(t, x)_{(2)} + D^+ V_2(t, x)_{(2)} - (M_1 + M_2) \| R(t, x(t)) \| \leq$$
$$\leq g_2(t, V_1(t, x) + V_2(t, x)) \leq g_2(t, m(t)) .$$

Theorem 1.4.1 of [3] gives for $t \in [t_2, t_1]$,

$$V_1(t, x(t, t_0, x_0)) + V_2(t, x(t, t_0, x_0)) \leq r_2(t \cdot t_2, v_0) + \tag{18}$$
$$+ (M_1 + M_2) \int_{t_2}^{t} \| R(s, x(s)) \| ds$$

where $r_2(t, t_2, v_0)$ is the maximal solution of (6) such that $r_2(t_2, t_2, v_0) = v_0$.

Also by assumption (i) and Theorem 1.4.1 of [3]

$$(19) \qquad V_1(t, x(t, t_0, u_0)) \leq r_1(t, t_0, u_0) \qquad \text{for} \quad t \geq t_0,$$

where $r_1(t, t_0, u_0)$ is the maximal solution of (5) such that $r_1(t_0, t_0, u_0) = u_0$. For $t \in [t_2, t_1]$ set $\phi(t) = (M_1 + M_2) \| R(t, x(t)) \|$, then

$$\int_{t_2}^{t_1} \phi(s)\, ds \leq \int_{t_2}^{t_1} \sup_{x \in \overline{S(\beta)}} (M_1 + M_2) \| R(s, x(s)) \|\, ds \leq (M_1 + M_2)\, \alpha \leq \alpha_1 .$$

Take $\bar{t} > t_1$ such that

$$\bar{t} - t_1 < \frac{2\left(\alpha_1 - \int_{t_2}^{t_1} \phi(s)\, ds\right)}{1 + \phi(t_1)}$$

and set $\phi(\bar{t}) = 0$, $\phi(t)$ to be linear on $[t_1, \bar{t}]$ and $\phi(t) = 0$ for $t \geq \bar{t}$. Then ϕ has a continuous extension for all $t \geq t_1$. Suppose $r^*(t, t_0, u_0)$ is the maximal solution of the perturbed differential equation (9), then by equi-integral stability of (5) it follows from $u_0 = V_1(t_0, x_0) < \alpha_1$ and (11) that $r^*(t, t_0, u_0) < \beta_1$ for $t \geq t_0$ and since $\lambda(t) = \phi(t)$ on $[t_2, t_1]$ we have $r^*(t, t_0, u_0) = r_1(t, t_0, u_0)$ on $[t_2, t_1]$. Clearly by (19),

$$V_1(t_2, t_0, V_1(t_0, x_0)) < \beta_1 .$$

Assumption (ii) and (17) imply $V_2(t_2, x(t_2, t_0, x_0)) \leq a(\alpha)$, so that

$$v_0 = V_1(t_2, x(t_2, t_0, x_0)) + V_2(t_2, t_0, x_0)) < \beta_1 + a(\alpha) \leq \alpha_2 .$$

Finally since $V_1 \geq 0$, (12), (13), (17) and assumption (ii) imply

$$\begin{aligned} b(\beta) &\leq r_2(t_1, t_2, v_0) + (M_1 + M_2) \int_{t_2}^{t_1} \| R(s, x(s)) \|\, ds \leq \\ &\leq \beta_2(\alpha_2) + (M_1 + M_2) \int_{t_2}^{t_1} \sup_{t \in \overline{S(\beta)}} \| R(s, x) \|\, ds \\ &\leq \beta_2(\alpha_2) + \alpha_2 < b(\beta) . \end{aligned}$$

which is a contraction.

If (16) holds, where $t_2 > t_0$ satisfies (17) and $u_0 = V_1(t^*, x(t^*, t_0, x_0))$ then since $x(t^*, t_0, x_0) \in \partial E$ and $V_1(t^*, x(t^*, t_0, x_0)) \leq \alpha^* < \alpha_2$, (19) implies $r_1(t_2, t^*, u_0) < \beta_1$ and again $v_0 < \beta_1 + a(\alpha) \leq \alpha_2$ and in this case

$$b(\beta) \leq r_2(t_1, t_2, v_0) + (M_1 + M_2) \int_{t_2}^{t_1} \| R(s, x(s)) \|\, ds \leq \beta_2(\alpha_2) + \alpha_2 < b(\beta).$$

If $\alpha < \rho$, then set $\beta(t_0, \alpha) = \beta(t_0, \rho)$ and proceed in a similar way. Hence the theorem.

Remarks. Our result shows that to obtain equi-integral stability of differential equations it is sufficient to assume conditions on our Lyapunov functions in the complement of a compact set in R^n. This has been achieved by perturbing the Lyapunov functions suitably. The proofs of known results on equi-integral stability of differential equations require that assumptions hold everywhere in R^n so our result improves the result of Theorem 3.9.1 of [3]. Moreover in the special case $g_1 = 0, g_2 = 0$, we also have an improvement on the result of Corollary 3.9.2 of [3].

The following theorem gives a set of sufficient conditions for the equi-asymptotic integral stability of (1) under assumptions weaker than known results [3].

THEOREM 2.5. *Assume that* (i) *and* (ii) *of Theorem 2.4 hold. Let the trivial solution of* (5) *be equi-asymptotically integrally stable and the trivial solution of* (6) *be uniform Lagrange stable. Then the trivial solution of* (1) *is equi-asymptotically integrally stable.*

Proof. By Theorem 2.4 the equi-integral stability of (1) follows. Let α and $\beta = \beta(t_0, \alpha)$ be as in Theorem 2.4. Then equi-asymptotic integral stability of (5) implies given $b(\varepsilon) > 0$, $\alpha_1 \geq 0$ and $t_0 \in R^+$, there exist $\gamma_1 = \gamma_1(t_0, \alpha_1, \varepsilon)$ and $T_1 = T_1(t_0, \alpha_1, \varepsilon)$ such that $\phi \in C(R^+, R^+)$ with

$$\int_0^\infty \phi(s)\, ds \leq \gamma_1, \tag{20}$$

every solution $u(t, t_0, u_0)$ of the perturbed scalar differential equation (9) satisfies

$$u(t, t_0, u_0) < b(\varepsilon) \qquad \text{for} \quad t \geq t_0 + T_1. \tag{21}$$

Set $a_3 = b(\varepsilon) + a(\alpha)$, then given $a_3 \geq 0$ and $t_0 \in R^+$, uniform Lagrange stability implies the existence of $T_1 = T_2(a_3, \varepsilon)$ such that $v_0 \leq \alpha_3$ implies

$$v(t, t_0, v_0) < \frac{b(\varepsilon)}{2} \qquad \text{for} \quad t \geq t_0 + T_2, \tag{22}$$

where $v(t, t_0, v_0)$ is any solution of the scalar differential equation (6). Let $T = \max\{T_1, T_2\}$ and choose a positive number $\gamma(t_0, \alpha, \varepsilon)$ so that $\gamma = \min\left\{\frac{b(\varepsilon)}{2(M_1+M_2)}, \frac{\gamma_1}{(M_1+M_2)}\right\}$. Then T and γ are the required positive numbers for the equi-asymptotic stability of (1). Suppose not; then there exists a sequence $\{t_k\}$ such that $t_k \geq t_0 + T$, $t_k \to \infty$ as $k \to \infty$ such that for any solution $x(t, t_0, x_0)$ of (2) with $\|x_0\| \leq \alpha$ and $\int_{t_0}^\infty \sup_{\|x\| \leq \beta} \|R(s, x(s))\|\, ds \leq \gamma$,

$$\|x(t_k, t_0, x_0)\| \geq \varepsilon. \tag{23}$$

Define for $t \geq t_0 + T$,

$$m(t) = V_1(t, x(t, t_0, x_0)) + V_2(t, x(t, t_0, x_0)) - (M_1 + M_2) \int_{t_0+T}^{t} \| R(s, x(s)) \| ds$$

and set $v_0 = V_1(t_0 + T, x(t_0 + T, t_0, x_0)) + V_2(t_0 + T, x(t_0 + T, t_0, x_0))$.

Then by (8), the monotone nonincreasing property of g_2 and Theorem 1.4.1 of [3] yields,

$$m(t) \leq r_2(t, t_0 + T, m(t_0 + T)) \quad \text{for} \quad t \geq t_0 + T, \tag{24}$$

where $r_2(t, t_0 + T, v_0)$ is the maximal solution of (6) such that $r_2(t_0 + T, t_0 + T, v_0) = v_0$. Set $u_0 = V_1(t_0 + T, x(t_0 + T, t_0, x_0))$, then assumption (i) and Theorem 1.4.1 of [3] imply

$$V_1(t, x(t, t_0, x_0)) \leq r_1(t, t_0, u_0) \qquad \text{for} \quad t \geq t_0 + T.$$

Now set $\phi(t) = (M_1 + M_2) \| R(t, x(t)) \|$, then

$$\int_{t_0}^{\infty} \phi(s) ds \leq \int_{t_0}^{\infty} (M_1 + M_2) \sup_{\|x\| \leq \beta} \| R(s, x(s) \| ds \leq (M_1 + M_2)\gamma \leq \gamma_1,$$

therefore $r_1(t, t_0, u_0) < b(\varepsilon)$ for $t \geq t_0 + T$. Moreover,

$$V_2(t_0 + T, x(t_0 + T, t_0, x_0)) \leq a(\|x\|) \leq a(\alpha),$$

so that

$$v_0 = V_1(t_0 + T, x(t_0 + T, t_0, x_0)) + V_2(t_0 + T, x(t_0 + T, t_0, x_0)) <$$
$$< b(\varepsilon) + a(\alpha) = \alpha_3.$$

Assumption (ii), (22), and (23) with $V_1 \geq 0$ imply,

$$b(\varepsilon) \leq V_1(t_k, x(t_k, t_0, x_0)) + V_2(t_k, x(t_k, t_0, x_0)),$$
$$\leq r_2(t_k, t_0 + T, v_0) + (M_1 + M_2) \int_{t_0+T}^{\infty} \| R(s, x(s)) \| ds <$$
$$< \frac{b(\varepsilon)}{2} + (M_1 + M_2)\gamma = b(\varepsilon),$$

which is a contradiction. The proof is complete.

Remarks. If instead of uniform boundedness of (6) we assume uniform integral stability of (6) and require $g_2 \in C(R^+ \times R^+, R)$; $g_2(t, 0) = 0$, in Theorem 2.4, the conclusion of that theorem remains valid. However if instead of uniform Lagrange stability of (6) we assume uniform integral stability in Theorem 2.5 the proof breaks down. The assumption of uniform asymptotic integral stability of (6) along with the above conditions on g_2 in Theorem 2.5

makes the conclusion of that Theorem to remain valid. The proof in that case may be constructed as in Theorem 2.5 with appropriate changes.

The following corollaries follow immediately from our theorems.

COROLLARY 2.6. *With the assumptions of Theorem 2.4, the uniform integral stability of the trivial solution of* (5) *and the uniform boundedness of* (6) *imply the uniform integral stability of the trivial solution of* (1).

COROLLARY 2.7. *If* $g_1 = 0, g_2 = 0$ *in Corollary 2.6, then the same conclusion holds.*

COROLLARY 2.8. *With the assumption of Theorem 2.4, the uniform asymptotic integral stability of* (5) *and the uniform Lagrange stability of* (6) *imply the uniform asymptotic integral stability of* (1).

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