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Periodic solutions of certain third order differential equations of the non-dissipative type

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Equazioni differenziali ordinarie. — *Periodic solutions of certain third order differential equations of the non-dissipative type.* Nota (*) di JAMES O. C. EZEILO, presentata dal Socio G. SANSONE.

RIASSUNTO. — Si dimostrano quattro teoremi per alcune classi di equazioni differenziali del terzo ordine di tipo non-dissipativo.

1. INTRODUCTION

Consider the linear third order differential equation:

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = p(t) \tag{1.1}$$

in which a, b, c are constants and p is a continuous function. It is well known that if the Routh-Hurwitz conditions:

$$a > 0 \quad , \quad b > 0 \quad , \quad ab > c > 0 \tag{1.2}$$

hold, the roots of the corresponding auxiliary equation:

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{1.3}$$

then have negative real parts so that the global asymptotic stability of solutions in the case $p \equiv 0$, the ultimate boundedness property of solution when $p\ (\not\equiv 0)$ is bounded, as well as the existence of ω-periodic solutions when p is also ω-periodic can all be very easily verified for (1.1) when (1.2) holds. Generalizations of all these results now abound extensively in the literature for several nonlinear third order differential equations (normally described as dissipative) in which a, b, c, not necessarily all constants, are subject to generalizations of (1.2) in some form or other. A fairly comprehensive bibliography of results in this area can be found in [1].

When (1.2) is not fulfilled, the existence of ω-periodic solutions (if p is ω-periodic) can still be established for a variety of equations (1.1) and generalizations of these to non-linear cases are also known, although the results here are much fewer than in the dissipative case. Some examples are listed in [1] but see also [2] and [3]. The object of the present paper is to give some other results in this " non Routh-Hurwitz " direction.

To be more specific let us take again the auxiliary equation (1.3). It is easy to verify that (1.3) has no purely imaginary root $\lambda = 2\pi i\omega^{-1}$ $(\omega > 0)$ if

$$ac < 0 , \qquad b \ \text{arbitrary} \tag{1.4}$$

(*) Pervenuta all'Accademia il 28 settembre 1977.

or if

$$(1.5) \qquad ac > 0 \quad \text{and} \quad a^{-1} c \neq 4 \pi^2 \omega^{-2}, \qquad b \text{ arbitrary}.$$

(Note here incidentally that by replacing t by $-t$ in (1.1) we may assume with respect to (1.5) that a, c are both positive). Thus, if p is ω-periodic in t, the linear equation (1.1) has indeed ω-periodic solutions if a, b, c are also subject to (1.4) or (1.5), and we shall see here that suitable extensions of this are also available for more general third order equations (1.1) in which a, b, c are not all necessarily constants. The results are given in Theorems 1, 2 and 3 which follow. We shall also show (in Theorem 4) that there are equations (1.1) with a as before but with b, c not constants and non Routh-Hurwitz for which ω-periodic solutions exist if p is ω-periodic in t.

2. STATEMENT OF RESULTS

We begin with the third order differential equation:

$$(2.1) \qquad \dddot{x} + f(\dot{x})\ddot{x} + g(x)\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x})$$

whre f, g, h, p are continuous functions depending only on the arguments shown and p is ω-periodic in t, that is $p(t, x, y, z) = p(t + \omega, x, y, z)$ $(\omega > 0)$ for arbitrary x, y, z, t. Our first result, in generalization of (1.4), is the following:

THEOREM 1. *Let* $\mathrm{F}(y) \equiv \int_0^y f(\eta)\, d\eta$ *and suppose that*

(i) *there exist positive constants* d_1, d_2 *such that*

$$(2.2) \qquad y\mathrm{F}(y) \leq -d_1 y^2 + d_2 \qquad \textit{for all } y;$$

(ii) *there exists a constant* E *such that*

$$(2.3) \qquad |p(t, x, y, z)| \leq \mathrm{E} \qquad \textit{for all } x, y, z, t;$$

(iii) *h satisfies*

$$(2.4) \qquad h(x) \operatorname{sgn} x \geq \mathrm{E} \qquad (|x| \geq 1).$$

Then (2.1) *has at least one* ω*-periodic solution, for all arbitrary* $g(x)$.

The situation corresponding to $a > 0$ and $c < 0$ is summarized in the following (with $\mathrm{F}(y)$ as before);

THEOREM 2. *Suppose that*

(i) *there exist positive constants* d_3, d_4 *such that*

$$(2.5) \qquad y\mathrm{F}(y) \geq d_3 y^2 - d_4 \qquad \textit{for all } y;$$

(ii) *p satisfies condition* (ii) *above in Theorem 1;*

$$h(x)\operatorname{sgn} x \leq -\mathrm{E} \qquad (|x| \geq 1). \tag{2.6}$$

Then (2.1) *has at least one* ω*-periodic solution, for all arbitrary* $g(x)$.

The only generalization of (1.5) which we have been able to achieve concerns third order equations of the form

$$\dddot{x} + \phi(\dot{x})\ddot{x} + \psi(\dot{x}) + \theta(x) = p(t, x, \dot{x}, \ddot{x}) \tag{2.7}$$

with ϕ, ψ, θ, p continuous and dependent only on the arguments shown and p ω-periodic in t as before. We shall however require here that $\theta'(x)$ exists and is continuous for all x.

We have

THEOREM 3. *Suppose that*

(i) *there are positive constants* a, c *with*

$$a^{-1} c < 4\pi^2 \omega^{-2} \tag{2.8}$$

such that

$$\theta'(x) \leq c \qquad \text{and} \quad \phi(y) \geq a \qquad \text{for all} \quad x, y; \tag{2.9}$$

(ii) *the function* θ *satisfies*

$$\theta(x)\operatorname{sgn} x \to +\infty \qquad \text{as} \quad |x| \to \infty \tag{2.10}$$

(iii) *the function* p *satisfies* (2.3).

Then (2.7) *has at least one* ω*-periodic solution, for all arbitrary* ψ.

It will have been noticed that the generalization here of (1.5) is only a partial one since it deals only with the aspect: $a^{-1} c < 4\pi^2 \omega^{-2}$. It will have been also of interest, for example, to establish the existence result subject to the conditions:

$$\phi(y) \leq a \quad , \quad \theta'(x) \geq c \quad , \quad a^{-1} c > 4\pi^2 \omega^{-2}$$

or some such conditions. All efforts in this direction for (2.7) have so far been unsuccessful. It is nevertheless important to stress that, so long as the insistence on *arbitrary* b remains, the condition $a^{-1} c \neq 4\pi^2 \omega^{-2}$ cannot be dispensed with as is seen by a consideration of the equation:

$$\dddot{x} + \ddot{x} + \dot{x} + x = \sin t$$

(corresponding to $a^{-1} c = 1 = 4\pi^2 \omega^{-2}$) which has no periodic solutions whatever.

We shall conclude with a consideration of the third order equation

$$\dddot{x} + a\ddot{x} + \beta(x)\dot{x} + \gamma(x) = p(t, x, \dot{x}, \ddot{x}) \tag{2.11}$$

in which $a > 0$ is a constant and β, γ, p are continuous functions depending only on the arguments shown, with p ω-periodic. The object here is to extend somewhat our earlier result in [2] by proving

THEOREM 4. *Suppose, given the equation* (2.11), *that there are constants* $E_1 > 0$, $E_2 > 0$ *such that*

(i) $|p(t, x, y, z)| \leq E_1 + E_2 (x^2 + y^2 + z^2)^{\frac{1}{2}}$ *for all* x, y, z, t

(ii) β *and* γ *are such that*

$$\beta(x) \leq b \quad \text{and} \quad x\gamma(x) \geq cx^2 - d \quad \text{for all } x, \tag{2.12}$$

where $b > 0$, $c > 0$ *and* $d \geq 0$ *are constants and*

$$ab < c. \tag{2.13}$$

Then there exists a constant $\varepsilon_0 > 0$ *whose magnitude depends only on* E_1, a, b, c *and* d *such that if* $E_2 \leq \varepsilon_0$ *then,* (2.11) *has at least one* ω*-periodic solution.*

3. SOME PRELIMINARY COMMENTS ON NOTATION

As in [2] the capitals $D, D_1, D_2, \cdots$ with or without suffixes will denote positive constants whose magnitudes depend only on the functions which occur in the specific differential equation under consideration. Thus, as an example, $D, D_1, D_2, \cdots$ which appear in § 4 below are constants whose magnitudes depend only on d_1, d_2, d_3, E, f, g and h. The numbering of the D's with suffixes will start afresh with each differential equation. Finally, as before, D's with suffixes are not necessarily the same in each place of occurrence, but the D's : $D_1, D_2, \cdots$ retain a fixed indentity throughout the proof of whatever equation is being considered.

4. PROOFS OF THEOREMS 1 AND 2

It will be enough to prove only Theorem 1, since the replacement of t by $-t$ in (2.1) when f and h are subject to (2.5) and (2.6) respectively yields a new equation (2.1) in which the corresponding f and h are now subject to to (2.2) and (2.4) respectively.

We therefore turn to (2.1) with f, p and h subject respectively to (2.2), (2.3) and (2.4) and with g an arbitrary function of x. Our proof is by the Leray-Schander technique, for which purpose we embed (2.1) in the parameter-dependent equation

$$\dddot{x} + \{\mu f(\dot{x}) - (1 - \mu) d_1\} \ddot{x} + \mu g(x) \dot{x} + (1 - \mu) c_1 x + \mu h = \mu p \tag{4.1}$$

where $c_1 > 0$ is an arbitrarily fixed constant and the parameter μ, as usual, satisfies $0 \leq \mu \leq 1$. The equation (4.1) can be reset in the form:

$$\dddot{x} - d_1 \ddot{x} + c_1 x + \mu \{(f(\dot{x}) + d_1) \ddot{x} + g(x) \dot{x} - c_1 x + h(x) - p\} = 0,$$

or, indeed, in the system form {with $X = \mathrm{col}(x, y, z)$ and $y \equiv \dot{x}$, $z \equiv \ddot{x}$};

$$\dot{X} = AX + \mu H(X, t) \tag{4.2}$$

where A, H are the matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -c_1 & 0 & d_1 \end{bmatrix} \quad H = \begin{pmatrix} 0 \\ 0 \\ -\{d_1 + f(y)\} z - y g(x) - h(x) + c_1 x + p(t, x, y, z) \end{pmatrix}$$

The eigenvalues of the matrix A here are the roots of the λ-equation

$$\lambda^3 - d_1 \lambda^2 + c_1 = 0 . \tag{4.3}$$

By comparison with (1.4), d_1 and c_1 here being positive, it follows that there are no purely imaginary roots $\lambda = 2\pi i \omega^{-1}$ for (4.3); and hence the matrix $(e^{-\omega A} - I)$ (I being the identity 3×3 matrix) is invertible and the procedure as outlined in [2; § 2] thus applies. In particular X is an ω-periodic solution of (4.2) if and only if

$$X = \mu TX \tag{4.4}$$

where

$$(TX)(t) = \int_t^{t+\omega} (e^{-\omega A} - I)^{-1} e^{(t-s)A} H(x(s), s) \, ds .$$

Thus the existence of an ω-periodic solution of (4.2), and therefore of (4.1) for all $\mu \in [0, 1]$ will follow for the usual reasons if the a priori bound:

$$\max_{0 \le t \le \omega} (|x(t)| + |\dot{x}(t)| + |\ddot{x}(t)|) \le D \tag{4.5}$$

can be established for every ω-periodic solution of (4.1) with $\mu \in (0, 1)$, which we now turn to establish.

Let then $x = x(t)$ be an arbitrary ω-periodic solution of (4.1) with $0 < \mu < 1$. Multiply both sides of (4.1) by x and integrate with respect to t. We have, since

$$\int x \dddot{x} \, dt = x\ddot{x} - \tfrac{1}{2} \dot{x}^2 \quad , \quad \int x\ddot{x} \, dt = x\dot{x} - \int \dot{x}^2 \, dt$$

$$\int x f(\dot{x}) \ddot{x} \, dt = x F(\dot{x}) - \int \dot{x} F(\dot{x}) \, dt$$

and x here is periodic, that

$$0 = -\mu \int_\tau^{\tau+\omega} \dot{x} F(\dot{x}) \, dt + (1 - \mu) d_1 \int_\tau^{\tau+\omega} \dot{x}^2 \, dt + (1 - \mu) c_1 \int_\tau^{\tau+\omega} x^2 \, dt + \\ + \mu \int_\tau^{\tau+\omega} x \{h(x) - p(t, x, \dot{x}, \ddot{x})\} \, dt . \tag{4.6}$$

But, by (2.2)

$$(4.7) \qquad -\mu \int_{\tau}^{\tau+\omega} \dot{x} F(\dot{x})\, dt + (1-\mu)\, d_1 \int_{\tau}^{\tau+\omega} \dot{x}^2\, dt \geq d_1 \int_{\tau}^{\tau+\omega} \dot{x}^2\, dt - D_1 .$$

Also, from (2.3) and (2.4), it is clear that

$$h(x) - p(t, x, \dot{x}, \ddot{x}) \geq 0 \quad \text{if } x \geq 1$$
$$\leq 0 \quad \text{if } x \leq -1$$

so that,

$$x\{h(x) - p(t, x, \dot{x}, \ddot{x})\} \geq 0 \quad \text{if } |x| \geq 1$$

whereas we have, h being continuous, that

$$|x\{h(x) - p(t, x, \dot{x}, \ddot{x})\}| \leq D \quad \text{if } |x| \leq 1 .$$

Thus, so long as $0 < \mu < 1$, (4.6) shows that

$$(4.8) \qquad \int_{\tau}^{\tau+\omega} \dot{x}^2\, dt \leq D_2 .$$

Next, direct integration (without any premultiplication) of (4.1) with respect to t yields:

$$(4.9) \qquad \int_{\tau}^{\tau+\omega} \{(1-\mu)\, c_1 x + \mu\, [h(x) - p(t, x, \dot{x}, \ddot{x})]\}\, dt = 0 .$$

From this it is easy to see that $|x(T)| < 1$ for some T. For otherwise we have that either $x(t) \geq 1$ for all t or $x(t) \leq -1$ for all t and in the former case the left hand side of (4.9) will be strictly positive and in the latter strictly negative, for $0 < \mu < 1$. Thus $|x(T)| < 1$ for some T, so that since

$$x(t) = x(T) + \int_{T}^{t} \dot{x}(s)\, ds$$

whe have that

$$(4.10) \qquad \max_{\tau \leq t \leq \tau+\omega} |x(t)| \leq 1 + \int_{T}^{T+\omega} |\dot{x}(s)|\, ds$$
$$\leq 1 + \omega^{\frac{1}{2}} \left(\int_{T}^{T+\omega} \dot{x}^2(s)\, ds \right)^{\frac{1}{2}}$$
$$\leq D_3 ,$$

by (4.8).

To complete the theorem it remains now to establish analogous estimates for $|\dot{x}|$ and $|\ddot{x}|$, where, as before, x is an ω-periodic solution of (4.1) with $\mu \in (0, 1)$. We move on now to tackle $|\dot{x}|$. For this set $y = \dot{x}$ and thus rewrite (4.1) in the form

$$\ddot{y} + f_\mu(y)\,\dot{y} + \mu g(x)\,y = Q \tag{4.11}$$

where

$$f\mu = \mu f(y) - (1 - \mu)\,d_1 \quad \text{and} \quad Q = \mu p - (1 - \mu)\,c_1 x - h.$$

Note that

$$|Q| \leq D \qquad \text{(by (2.3) and (4.10))}. \tag{4.12}$$

Now multiply (4.11) by y and integrate. We obtain, y being ω-periodic, that

$$-\int_\tau^{\tau+\omega} \dot{y}^2\, dt = \int_\tau^{\tau+\omega} \{yQ - \mu g(x)\,y^2\}\, dt, \tag{4.13}$$

so that, since $|g(x)| \leq D$, by (4.10), we have, on using (4.12), the estimate:

$$\begin{aligned} \int_\tau^{\tau+\omega} \dot{y}^2\, dt &\leq D \int_\tau^{\tau+\omega} (|y| - y^2)\, dt \\ &\leq D \left\{ \left(\int_\tau^{\tau+\omega} y^2\, dt \right)^{\frac{1}{2}} + \int_\tau^{\tau+\omega} y^2\, dt \right\} \\ &\leq D_4 \end{aligned} \tag{4.14}$$

by (4.8). Now, since $x(t) = x(t + \omega)$, it is clear that $y(T) \equiv \dot{x}(T) = 0$ for some T. Hence, because of the identity:

$$y(t) = y(T) + \int_T^t \dot{y}(s)\, ds,$$

we have htat

$$\begin{aligned} \max_{T \leq t \leq T+\omega} |\dot{x}(t)| &\leq \left| \int_T^{T+\omega} \dot{y}(s)\, ds \right| \\ &\leq \omega^{\frac{1}{2}} \left(\int_T^{T+\omega} \dot{y}^2(s)\, ds \right)^{\frac{1}{2}} \\ &\leq D_5 \end{aligned} \tag{4.15}$$

by (4.14).

We come finally to the a priori boundedness of $\ddot{x}(t)$. For this multiply (4.11) by $\ddot{y}$ and integrate. We have that

$$\int_{\tau}^{\tau+\omega} \ddot{y}^2\, at = -\int_{\tau}^{\tau+\omega} f_{\mu}(y)\, \dot{y}\ddot{y}\, dt - \int_{\tau}^{\tau+\omega} \{\mu g(x)\, y - Q\}\, \ddot{y}\, dt$$

so that, by (4.10), (4.12) and (4.15),

$$(4.16) \qquad \int_{\tau}^{\tau+\omega} \ddot{y}^2\, dt \leq D \int_{\tau}^{\tau+\omega} |\dot{y}|\, |\ddot{y}|\, dt + D \int_{\tau}^{\tau+\omega} |\ddot{y}|\, dt$$

$$\leq D \left(\int_{\tau}^{\tau+\omega} \dot{y}^2\, dt\right)^{\frac{1}{2}} \left(\int_{\tau}^{\tau+\omega} \ddot{y}^2\, dt\right)^{\frac{1}{2}} + D \left(\int_{\tau}^{\tau+\omega} \ddot{y}^2\, dt\right)^{\frac{1}{2}}$$

$$\leq D \left(\int_{\tau}^{\tau+\omega} \ddot{y}^2\, dt\right)^{\frac{1}{2}},$$

by (4.14). The estimate (4.16) shows clearly that

$$(4.17) \qquad \int_{\tau}^{\tau+\omega} \ddot{y}^2\, dt \leq D\,.$$

and by using the fact that, $y(t)$ being ω-periodic, $\ddot{x}(T) \equiv \dot{y}(T) = 0$ for some T, so that $\ddot{x}(t) = \int_{T}^{t} \ddot{y}(s)\, ds$ it can now be shown in the same way as before from (4.17) that

$$(4.18) \qquad \max_{\tau \leq t \leq \tau+\omega} |\ddot{x}(t)| \leq D\,.$$

The estimates (4.10), (4.15) and (4.18) clearly establish (4.5) and hence also Theorem 1.

This also concludes the verification of Theorem 2 for the reasons pointed out at the beginning of this paragraph.

5. PROOF OF THEOREM 3

The procedure and the major tools here are as in § 4 and we shall therefore skip inessential details.

We work here with a parameter-dependent equation (analagous to (4.1)) of the form:

$$(5.1) \qquad \dddot{x} + \{(1-\mu)\, a + \mu\phi(\dot{x})\}\, \ddot{x} + \mu\psi(\dot{x}) + (1-\mu)\, cx + \mu\theta(x) = \\ = \mu p(t, x, \dot{x}, \ddot{x})$$

wich, when $\mu = 1$, reduces to (2.1) and when $\mu = 0$, to the equation

$$\dddot{x} + a\ddot{x} + cx = 0$$

with an auxilliary equation

$$\lambda^3 + a\lambda^2 + c = 0$$

which has no purely imaginary roots of the form $\lambda = 2\pi i \omega^{-1}$ because of (2.8) and (1.5). Thus the selection of the parameter-dependent equation is in order with respect to the property of (5.1) at $\mu = 0$. To complete the proof of Theorem 3 then it remains only to establish the a priori bound (4.5) for every ω-periodic solution of (5.1) with $0 < \mu < 1$.

Let then $x = x(t)$ be an ω-periodic solution of (5.1) $(0 < \mu < 1)$. Multiply both sides of (5.1) by $\ddot{x}$ and integrated. We have

$$(5.2) \qquad \int_\tau^{\tau+\omega} \{(1-\mu)\, a + \mu\phi(\dot{x})\}\, \ddot{x}^2 \, dt + \int_\tau^{\tau+\omega} \{(1-\mu)\, cx + \mu\theta(x)\}\, \ddot{x} \, dt =$$

$$= \mu \int_\tau^{\tau+\omega} p(t, x, \dot{x}, \ddot{x})\, \ddot{x} \, dt$$

By (2.9)

$$(1-\mu)\, a + \mu\phi(\dot{x}) \le a .$$

Also, since

$$\int x\ddot{x} \, dt = x\dot{x} - \int \dot{x}^2 \qquad \text{and} \qquad \int \theta(x)\, \ddot{x} \, dt = \dot{x}\theta(x) - \int \theta'(x)\, \dot{x}^2 \, dt ,$$

$$\int_\tau^{\tau+\omega} \{(1-\mu)\, cx + \mu\theta(x)\}\, \ddot{x} \, dt = -\int_\tau^{\tau+\omega} \{(1-\mu)\, c + \theta'(x)\}\, \dot{x}^2 \, dt$$

$$\ge -c \int_\tau^{\tau+\omega} \dot{x}^2 \, dt$$

by (2.9). Thus (5.2) gives that

$$(5.3) \qquad a \int_\tau^{\tau+\omega} \ddot{x}^2 \, dt - c \int_\tau^{\tau+\omega} \dot{x}^2 \, dt \le E \int_\tau^{\tau+\omega} |\ddot{x}| \, dt .$$

Now, if x has the Fourier expansion

$$x \sim \sum_{r=0}^{\infty} (a_r \cos 2\pi r\omega^{-1} t + b_r \sin 2\pi r\omega^{-1} t)$$

then $\dot{x}$ and $\ddot{x}$ have the expansions:

$$\dot{x} \sim 2\,\pi\omega^{-1} \sum_{r=1}^{\infty} (-r a_r \sin 2\,\pi r\omega^{-1} + r b_r \cos 2\,\pi\, r\omega^{-1}\, t)$$

$$\ddot{x} \sim -4\,\pi^2\,\omega^{-2} \sum_{r=1}^{\infty} (r^2\, a_r^2 \cos 2\,\pi r\omega^{-1} t + r^2\, b_r \sin 2\,\pi r\omega^{-1} t)$$

and a straightforward consideration of the last two expansions will verify in the usual way that

$$\int_{\tau}^{\tau+\omega} \ddot{x}^2\, \mathrm{d}t \geq 4\,\pi^2\,\omega^{-2} \int_{\tau}^{\tau+\omega} \dot{x}^2\, \mathrm{d}t\,.$$

Hence, by (5.3)

$$\left(a - \frac{1}{4}\, c\omega^2\, \pi^{-2}\right) \int_{\tau}^{\tau+\omega} \ddot{x}^2\, \mathrm{d}t \leq \mathrm{E} \int_{\tau}^{\tau+\omega} |\, \ddot{x}\, |\, \mathrm{d}t\,, \tag{5.4}$$

so that since the coefficient $\left(a - \frac{1}{4}\, c\omega^2\, \pi^{-2}\right)$ here is positive (by (2.8)) we also have from (5.4), and analagous to (4.14) that

$$\int_{\tau}^{\tau+\omega} \ddot{x}^2\, \mathrm{d}t \leq \mathrm{D}_1\,. \tag{5.5}$$

From this it is easy to establish, as before, that

$$\max_{\tau \leq t \leq \tau+\omega} |\, \dot{x}\, (t)\, | \leq \mathrm{D}_2\,. \tag{5.6}$$

For the uniform a priori boundedness of $|\, x\, (t)\, |$ we shall also, as before, realy on a direct integration of (5.1) which now gives that

$$\int_{\tau}^{\tau+\omega} -\{(1-\mu)\, cx + \mu\theta\, (x) - \mu p\}\, \mathrm{d}t = \mu \int_{\tau}^{\tau+\omega} \psi\, (\dot{x})\, \mathrm{d}t$$

or indeed, in view of (5.6), that

$$\left| \int_{\tau}^{\tau+\omega} \{(1-\mu)\, cx + \mu\theta\, (x) - \mu p\}\, \mathrm{d}t \right| \leq \mathrm{D}_3\,. \tag{5.7}$$

Since $c > 0$, $|\, p\, | \leq \mathrm{E} < \infty$ and $\theta\, (x) \operatorname{sgn} x \to +\infty$ as $|\, x\, | \to \infty$, it is clear that there exists D_4 independent of μ such that

$$|\, \{(1-\mu)\, cx + \mu\theta\, (x) - \mu p\}\, | \geq 2\, \mathrm{D}_3\, \omega^{-1} \tag{5.8}$$

if $|x(t)| \geq D_4$ for all $t \in [\tau, \tau + \omega]$. It is thus clear that $|x(T)| \leq D_4$ for some T, as otherwise, by (5.8), the left handside of (5.7) would be not less in magnitude than $2 D_3$. The result that $|x(T)| \leq D_4$ for some T combined with (5.6) to yield the required boundedness estimate for x:

$$\max_{\tau \leq t \leq \tau+\omega} |x(t)| \leq D_3 + D_2 \omega . \tag{5.9}$$

The boundedness of the remaining term $\ddot{x}$ can also be handled as in § 4 by multiplying both sides of (5.1) by $\dddot{x}$ and integrating. We obtain from the integration that

$$\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt = -\mu \int_{\tau}^{\tau+\omega} \phi(\dot{x}) \ddot{x}\dddot{x} \, dt - \int_{\tau}^{\tau+\omega} \{\mu\Psi(\dot{x}) + (1-\mu) cx + \mu\theta(x) - \mu p\} \, dt. \tag{5.10}$$

But by (5.6), $|\phi(\dot{x})| \leq D$ so that

$$\left| -\mu \int_{\tau}^{\tau+\omega} \phi(\dot{x}) \ddot{x}\dddot{x} \, dt \right| \leq D \int_{\tau}^{\tau+\omega} |\ddot{x}| \, |\dddot{x}| \, dt$$

$$\leq D \left(\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \right)^{\frac{1}{2}}$$

by (5.5). Next the term inside the curly bracket in the second integral on the right handside of (5.10) is bounded (by a D) in view of (5.6), (5.9) and (2.3). Thus we have, as before, from (5.10) that

$$\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \leq D \left(\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \right)^{\frac{1}{2}} + D$$

analagous to (4.16) from which the boundedness of $\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt$ and therefore the required boundedness estimate (4.18) for $\ddot{x}$ can now be obtained as before.

This completes our verification of Theorem 3.

6. Proof of Theorem 4

The procedure here is almost exactly as in [2] and I shall therefore only sketch the outlines.

It is convenient to take the parameter-dependent equation in the form:

$$\dddot{x} + a\ddot{x} + \{(1-\mu) b + \mu\beta(x)\} \dot{x} + \{(1-\mu) cx + \mu\gamma(x)\} = \mu p \tag{6.1}$$

which reduces to (2.11) when $\mu = 1$ and to

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0 \tag{6.2}$$

when $\mu = 0$ with the condition $c > ab$. The auxilliary equation corresponding to (6.2) has no purely imaginary roots and so the arguments in [2, § 2] are applicable without any modification whatever and all that remains, as shown there, in order to prove the theorem is to establish once again that (4.5) holds for any ω-periodic solution of (6.1) with $0 < \mu < 1$. For this it is vital to be able to show that if E_2 is sufficiently small then

$$\int_{\tau}^{\tau+\omega} (x^2 + \dot{x}^2 + \ddot{x}^2)\, dt \leq D \tag{6.3}$$

for any ω-periodic solution $x(t)$ whatever of (6.1) with $0 < \mu < 1$. The result (6.3) is the analogue of an inequality in the Lemma in [2; § 3] on which the proof of the a priori estimate (4.5) hinges.

Consider now the function $V = V(x, y, z)$ defined by

$$V = axz - a^2 xy - \delta yz$$

where $\delta > 0$ is a constant fixed, as is possible in view of (2.13), such that

$$a^2 b^{-1} > \delta > a^3 c^{-1}. \tag{6.4}$$

Let now $x = x(t)$ be an arbitrary ω-periodic solution of (6.1). A straightforward differentiation (with respect to t) of $V = V(x, \dot{x}, \ddot{x})$, with $\dddot{x}$ replaced by $-\{a\ddot{x} + \beta_\mu(x)\dot{x} + \gamma_\mu(x) - \mu p\}$ where

$$\beta_\mu(x) \equiv (1-\mu)\, b + \mu\beta(x) \quad , \quad \gamma_\mu(x) \equiv (1-\mu)\, cx + \mu\gamma(x),$$

will verify that

$$\dot{V} = U_1 - U_2 + \mu(ax - \delta\dot{x})\, p$$

where

$$U_1 = a\dot{x}\ddot{x} - ax\,\beta_\mu(x)\,\dot{x} + a\delta\,\dot{x}\ddot{x}$$

$$\begin{aligned} U_2 &= \delta\ddot{x}^2 + 2\,a^2 x\ddot{x} + ax\gamma_\mu(x) + [a^2 - \delta\beta_\mu(x)]\,\dot{x}^2 \\ &\equiv \delta(\ddot{x} + a^2\delta^{-1}x)^2 + a\,[x\gamma_\mu(x) - a^3\delta^{-1}x^2] + [a^2 - \delta\beta_\mu(x)]\,\dot{x}^2 . \end{aligned}$$

Now, by (2.12),

$$\beta_\mu(x) \leq b \qquad \text{and} \quad x\gamma_\mu(x) \geq cx^2 - d$$

for arbitrary $\mu \varepsilon (0, 1)$, so that

$$\begin{aligned} U_2 &\geq \delta(\ddot{x} + a^2\delta^{-1}x)^2 + a(c - a^3\delta^{-1})\,x^2 + (a^2 - \delta b)\,\dot{x}^2 - ad \\ &\geq D_1(x^2 + \dot{x}^2 + \ddot{x}^2) - ad , \end{aligned}$$

for some D_1, since $(c - a^3 \delta^{-1})$ and $(a^2 - \delta b)$ are both positive, by (6.4). Also, since

$$|p(t, x, y, z)| \leq E_1 + E_2 (x^2 + y^2 + z^2)^{\frac{1}{2}},$$

we have that

$$|\mu (ax - \delta \dot{x}) p(t, x, \dot{x}, \ddot{x})| \leq D_2 E_3 (x^2 + \dot{x}^2 + \ddot{x}^2) + D_3 (x^2 + \dot{x}^2)^{\frac{1}{2}}$$

for some D_2, D_3. Hence

$$\dot{V} \leq U_1 - (D_1 - D_2 E_3)(x^2 + \dot{x}^2 + \ddot{x}^2) + D_3 (x^2 + \dot{x}^2)^{\frac{1}{2}} + D .$$

Thus, if $E_2 \leq \frac{1}{2} D_1 D_2^{-1}$, wich we suppose, then

$$\dot{V} \leq U_1 - \tfrac{1}{2} D_1 (x^2 + \dot{x}^2 + \ddot{x}^2) + D_4 [(x^2 + \dot{x}^2)^{\frac{1}{2}} + 1] \tag{6.6}$$

from which, by integration, $x(t)$ being ω-periodic and U_1 a perfect differential, it follows that

$$\int_{\tau}^{\tau+\omega} (x^2 + \dot{x}^2 + \ddot{x}^2)\, dt \leq 2 D_1^{-1} D_4 \int_{\tau}^{\tau+\omega} \{(x^2 + \dot{x}^2)^{\frac{1}{2}} + 1\}\, dt$$

which in turn implies the required result (6.3).

The rest of the proof, progressively, from (6.3), of the boundedness of $|x(t)|$, $|\dot{x}(t)|$ and $|\ddot{x}(t)|$ now follows precisely as in [2; § 4] and further details will therefore be omitted.

This concludes our verification of Theorem 4.

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