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Periodic solutions of a certain fourth order differential equation

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Equazioni differenziali ordinarie. — *Periodic solutions of a certain fourth order differential equation.* Nota (*) di JAMES O. C. EZEILO, presentata dal Socio G. SANSONE.

RIASSUNTO. — Si dimostrano teoremi di esistenza di soluzioni periodiche per due classi di equazioni differenziali ordinarie non lineari.

1. INTRODUCTION

Consider the equation:

$$x^{(4)} + a_1 \dddot{x} + g(\dot{x})\ddot{x} + a_3\dot{x} + h(x, \dot{x}, \ddot{x}, \dddot{x}) = p(t) \tag{1.1}$$

in which a_1, a_3 are constants; g, h and p are continuous functions depending only on the arguments shown, with p ω-periodic in t, that is $p(t+\omega) = p(t)$ for some $\omega > 0$ and for arbitrary t. The existence of an ω-periodic solution of (1.1) for the special case in which h is bounded, that is $|h(x, y, z, u)| \leq H$ (constant) for arbitrary x, y, z and u, has attracted the interest of researchers in recent times. Tejumola [2] for example, generalizing an earlier result of himself [1] for the same equation (1.1), showed that if a_1, a_3 are *both* positive and if the following conditions are satisfied:

$$h(x, y, z, u) \operatorname{sgn} x > 0 \quad , \quad (|x| \geq 1), \tag{1.2}$$

$$\left(\int_0^y g(n)\,d\eta - a_1^{-1} a_3 y\right) \operatorname{sgn} y \to \infty \quad \text{as} \quad |y| \to \infty, \tag{1.3}$$

$$\left|\int_0^t p(s)\,ds\right| \leq a_0 \quad \text{(constant)} \qquad \text{for all} \quad t, \tag{1.4}$$

then (1.1) admits of, at least, one ω-periodic solution. Note that, because of the ω-periodicity of p, the condition (1.4) is equivalent to the following:

$$\int_0^\omega p(s)\,ds = 0. \tag{1.5}$$

Also, by replacing t by $-t$ in (1.1), it is clear that one can assume a_1, a_3 *both* negative so that Tejumola's existence result can, more generally, be said to hold for (1.1), with g, h and p subject to (1.2), (1.3) and (1.4) (or (1.5)), if $a_1 a_3 > 0$.

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The question arising therefrom is whether ω-periodic solutions of (1.1) exist when $a_1 a_3 \leq 0$ and under what conditions on g, p and h remaining as before. It turned out from our investigation that the treatment is indeed extendable to the more general equation:

$$x^{(4)} + a_1 \dddot{x} + g(\dot{x})\ddot{x} + \gamma(x)\dot{x} + h(x, \dot{x}, \ddot{x}, \dddot{x}, t) = p(t) \tag{1.6}$$

in which $\gamma(x)$ is a continuous function depending only on x and h, which depends (continuously) on all the arguments shown (which now include t), is ω-periodic in t (that is: $h(x, y, z, u, t+\omega) = h(x, y, z, u, t)$ for arbitrary x, y, z, u, t) and is bounded, as before, that is:

$$|h(x, y, z, u, t)| \leq \mathrm{H} \text{ (constant)} \quad \text{for all} \quad x, y, z, u, t. \tag{1.7}$$

Our answer to the question is summed up in the following

THEOREM 1. *Given that* (1.5) *and* (1.7) *hold, suppose that* $a_1 \neq 0$ *and that*

(i) *there is a constant* δ, *with* $0 \leq \delta < 4\pi^2\omega^{-2}$, *such that*

$$a_1^{-1}\gamma(x) \leq \delta \quad \textit{for all} \quad x, \tag{1.8}$$

(ii) *h satisfies the condition*

$$h(x, y, z, u, t) \operatorname{sgn} x \geq 0 \qquad (|x| \geq 1). \tag{1.9}$$

Then the equation (1.6) *admits of at least one ω-periodic solution for all arbitrary continuous* $g(\dot{x})$.

Observe that the lower bound restriction on $h \operatorname{sgn} x$ here in (1.9) is weaker than the corresponding one in (1.2) used by Tejumola in [1] and [2].

We have also, while at it, looked at the equation (1.6) outside the context of a mere generalization of the condition: $a_1 a_3 \leq 0$ for the equation (1.1), and one other existence result which we were able to establish, with practically the same tools as for Theorem 1, is the following:

THEOREM 2. *Given that* (1.5), (1.7) *and* (1.9) *hold, suppose that,* $a_1 \neq 0$ *and that*

$$|\Gamma(x)| \equiv \left| \int_0^x \gamma(s)\, ds \right| \leq a_2 \quad \textit{for all} \quad x \tag{1.10}$$

where a_2 is a constant. Then the equation (1.6) *admits of at least one ω-periodic solution for all arbitrary continuous* $g(\dot{x})$.

Note that, although hypothesis (i) of Theorem 1 has effectively covered the case $a_1 a_3 \leq 0$ which was the starting point of the problem the provisions of the hypothesis actually go beyond that context to show that there are cases $\gamma \equiv a_3$ (constant) with $a_1 a_3 > 0$ dealt with in [1], [2] for which the condition (1.3) on g is quite superfluous. For example, if $\gamma \equiv a_3$ in (1.6) with $0 < a_3 a_1^{-1} < 4\pi\omega^{-2}$ hypothesis (i) of Theorem 1 would be fully met and the existence result would therefore hold for *all arbitrary* $g(\dot{x})$. Again, while hypothesis (i) of Theorem 1 imposes a requirement that $\gamma(x)$ be bounded above or below according as $a_1 > 0$ or < 0, a consideration of the function $\gamma \equiv x \cos x^2$ which satisfies (1.10) (since $\Gamma(x) = \frac{1}{2}\sin x^2$) of Theorem 2 shows that there are nevertheless cases of (1.6) with γ not bounded (above or below) for which ω-periodic solutions exist for all arbitrary $g(\dot{x})$. It should be finally remarked that the condition $a_1 \neq 0$ cannot in general be dropped in either theorem as is best illustrated by a consideration of the equation

$$x^{(4)} + \ddot{x} = \text{cost}$$

(corresponding to $a_1 = 0$, $\gamma \equiv 0$, $g \equiv 1$ and $h \equiv 0$ in (1.6)) which has no periodic solutions whatever.

2. Comments on the procedure

The proof of either theorem will be by the Leray-Schauder fixed point technique, just as in [1] and [2] except that here it will be convenient to take, for our parameter-dependent equation, the equation:

$$x^{(4)} + a_1 \dddot{x} + \mu g(\dot{x})\ddot{x} + \mu\gamma(x)\dot{x} + (1-\mu)\, a_4 x + \mu h(x, \dot{x}, \ddot{x}, \dddot{x}, t) = \mu p(t) \tag{2.1}$$

in which a_4 is an arbitrarily chosen, but fixed, positive constant. We note that, as is usual in these cases, the parameter-dependent equation (2.1) reduces to (1.6) when the parameter $\mu = 1$ and to a constant-coefficient equation namely:

$$x^{(4)} + a_1 \dddot{x} + a_4 x = 0 \tag{2.2}$$

when $\mu = 0$. The equation (2.1) may also be taken in the system form:

$$\dot{X} = AX + \mu F(X, t) \tag{2.3}$$

with the 4-vectors X, F and 4×4 matrix A defined by:

$$X = \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \psi \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & 0 & 0 & -a_1 \end{pmatrix}$$

where $y = \dot{x}$, $z = \ddot{x}$, $u = \dddot{x}$ and

$$\psi = \psi(x, y, z, u, t) \equiv a_4 x - y\gamma(x) - zg(y) - h(x, y, z, u, t) + p(t).$$

The eigenvalues of A can be verified to be the roots of the auxiliary equation of (2.2) namely:

$$\gamma^4 + a_1\gamma^3 + a_4 = 0 \tag{2.4}$$

with $a_4 > 0$ it is clear that (2.4) has no roots whatever of the form $\gamma = i\beta$ (β real) and so the matrix $(e^{-\omega A} - I)$, I being the identity 4×4 matrix, is clearly invertible and so by adapting an argument in [4; 27–28] it is easily deduced that $X = X(t)$ is an ω-periodic solution of (2.3) if and only if X satisfies the equation

$$X = \mu TX \tag{2.5}$$

where

$$(TX)(t) = \int_t^{t+\omega} (e^{-\omega A} - I)^{-1} e^{(t-s)A} F(X(s), s)\, ds . \tag{2.6}$$

The situation here is thus exactly as in [3; § 2]., except that the vectors here are 4-vectors and the matrices 4×4 matrices, and the same arguments indeed as in [3] can be extended to show that the existence of an ω-periodic solution of (1.6) will follow if an *a priori* bound (in the standard uniform norm) *independent* of μ can be established for *all* ω-periodic solutions X of (**2.5**) with $\mu \in (0, 1)$, that is

$$\| X \| \leq D \tag{2.7}$$

with D independent of $\mu \in (0, 1)$. We shall actually here stop merely with verifying that there exists a constant D independent of μ such that, for *any* ω-periodic solution $X = \operatorname{col}(x, y, z, u)$ of (2.5),

$$\max | x(t) | \leq D \quad , \quad \max | y(t) | \leq D \quad \text{and} \quad \max | z(t) | \leq D \qquad (\tau \leq t \leq \tau + \omega) \tag{2.8}$$

for some τ. For suppose indeed that (2.8) holds, then because of (1.7) the last entry in the definition of the vector F is bounded for all $t \in [\tau, \tau + \omega]$; and therefore because of the definition (2.6) the vector TX is bounded (in the usual norm) by a constant independent of μ whose magnitude depends on D, so that since, anyway, $\| X \| \leq \| TX \|$ for all solutions of (2.5) with $\mu \in (0, 1)$ the required a priori bound (2.7) follows if (2.8) holds.

Summing up then, it is now clear from the foregoing that for our proof of either theorem it will suffice to concentrate on the equation (2.1) and to prove simply that there exists a constant $D > 0$ independent of μ such that

$$| x(t) | \leq D \quad , \quad | \dot{x}(t) | \leq D \quad , \quad | \ddot{x}(t) | \leq D \qquad (\tau \leq t \leq \tau + \omega) \tag{2.9}$$

for *any ω-periodic solution* of (2.1) with $\mu \in (0, 1)$.

3. COMMENTS ON THE NOTATION

Let $a_5 \equiv \max\limits_{0 \leq t \leq \omega} | p(t) |$.

In what follows the capitals $D, D_1, D_2, \cdots$ denote positive constants whose magnitudes depend only on $a_0, a_1, a_2, a_4, a_5, H, \gamma$ and g but certainly *not* on μ. Unnumbered D's do not always have the same value in each place of occurence, but each of the numbered D*s* : $D_1, D_2, \cdots$ retains a fixed indentity throughout.

4. Proof of Theorem 1

Let $x = x(t)$ be any ω-periodic solution of (2.1). It is convenniet to start by establishing for x the middle inequality in (2.9).

For this we will require the result

$$4\pi^2 \int_\tau^{\tau+\omega} \dot{x}^2 \, dt \leq \omega^2 \int_\tau^{\tau+\omega} \ddot{x}^2 \, dt \tag{4.1}$$

which derives exclusively from the ω-periodicity of x and the usual consideration of the Fourier series expansions of $\dot{x}$ and $\ddot{x}$.

The key however to our estimate of $|\dot{x}(t)|$ is the direct result of multiplying both sides of (2.1) by $\dot{x}$ and integrating with respect to t. Since

$$\int \dot{x}x^{(4)} \, dt = \dot{x}\dddot{x} - \tfrac{1}{2}\ddot{x}^2 \quad , \quad \int \dot{x}\dddot{x} \, dt = \dot{x}\ddot{x} - \int \ddot{x}^2 \, dt$$

$$\int x\dot{x} \, dt = \tfrac{1}{2}x^2 \quad , \quad \frac{d}{dt}\int_0^{\dot{x}} \eta g(\eta) \, d\eta = \dot{x}g(\dot{x})\,\ddot{x} \, ,$$

and x is ω-periodic, the integration shows at once that

$$\int_\tau^{\tau+\omega} \ddot{x}^2 \, dt - \mu a_1^{-1} \int_\tau^{\tau+\omega} \gamma(x)\,\dot{x}^2 \, dt = \mu a_1^{-1} \int_\tau^{\tau+\omega} (h - p)\,\dot{x} \, dt \, . \tag{4.2}$$

Now, by

$$- a_1^{-1} \int_\tau^{\tau+\omega} \gamma(x)\,\dot{x}^2 \, dt \geq -\delta \int_\tau^{\tau+\omega} \dot{x}^2 \, dt$$

$$\geq -\frac{1}{4}\,\delta\omega^2\pi^2 \int_\tau^{\tau+\omega} \ddot{x}^2 \, dt \, ,$$

by (4.1). Thus we have from (4.2), with $\mu \in (0, 1)$, h and p being bounded, that

$$\left(1 - \frac{1}{4}\,\delta\omega^2\pi^{-2}\right) \int_\tau^{\tau+\omega} \ddot{x}^2 \, dt \leq D \int_\tau^{\tau+\omega} |\dot{x}| \, dt$$

$$\leq D \left(\int_\tau^{\tau+\omega} \dot{x}^2 \, dt\right)^{\frac{1}{2}} ,$$

by Schwarz's inequality, so that, since $\delta < 4\pi^2\omega^{-2}$,

$$\int_\tau^{\tau+\omega} \ddot{x}^2 \, dt \leq D \left(\int_\tau^{\tau+\omega} \dot{x}^2 \, dt\right)^{\frac{1}{2}}$$

and therefore, by (4.1),

$$\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \leq D \left(\int_{\tau}^{\tau+\omega} \ddot{x}^2 \, dt \right)^{\frac{1}{2}} . \tag{4.3}$$

Hence

$$\int_{\tau}^{\tau+\omega} \ddot{x}^2 \, dt \leq D . \tag{4.4}$$

Now, since $x(0) = x(\omega)$, it is clear that $\dot{x}(\tau_0) = 0$ for some $\tau_0 \in [0, \omega]$. Thus

$$\begin{aligned} \dot{x}(t) &\equiv \dot{x}(\tau_0) + \int_{\tau_0}^{t} \ddot{x}(s) \, ds \\ &= \int_{\tau_0}^{t} \ddot{x}(s) \, ds \end{aligned}$$

and therefore

$$\begin{aligned} \max_{0 \leq t \leq \omega} |\dot{x}(t)| &\leq \int_{\tau_0}^{\tau_0+\omega} |\ddot{x}(s)| \, ds \\ &\leq \omega^{\frac{1}{2}} \left(\int_{\tau}^{\tau+\omega} \ddot{x}^2(s) \, ds \right)^{\frac{1}{2}} , \end{aligned}$$

by Schwarz's inequality. Hence

$$\max_{0 \leq t \leq \omega} |\dot{x}(t)| \leq D_1 , \tag{4.5}$$

by (4.4)

We are in a position to tackle the first estimate in (2.9). The main tool here is the result:

$$\int_0^{\omega} \{(1 - \mu) a_4 x + \mu h(x, \dot{x}, \ddot{x}, \dddot{x}, t)\} \, dt = 0 \qquad (0 < \mu < 1) \tag{4.6}$$

obtained from (2.1) by integrating directly and using (1.5) as well as the ω-periodicity of x. It is not difficult to see from (4.6) that $|x(\tau_1)| < 1$ for some $\tau_1 \in [0, \omega]$. For, in the contrary case: $|x(t)| \geq 1$ for all t, we would have by (1.9) that the left handside of (4.6) is strictly non zero for $0 < \mu < 1$ in manifest contradiction of (4.6) itself. Hence $|x(\tau_1)| < 1$ for some $\tau_1 \in [0, \omega]$ and thus, since

$$x(t) \equiv x(\tau_1) + \int_{\tau_1}^{t} \dot{x}(s) \, ds ,$$

we have that

$$(4.7) \qquad \max |x(t)| < 1 + \int_{\tau_1}^{\tau_1+\omega} |\dot{x}(s)| \, ds \leq 1 + D_1 \omega$$

by (4.5).

It remains now to establish the last inequality in (2.9). For this purpose let us set (2.1) in the form

$$(4.8) \qquad x^{(4)} + a_1 \ddot{x} = Q$$

where

$$Q = \mu p - \mu g(\dot{x}) \ddot{x} - \mu \gamma(x) \dot{x} - (1 - \mu) a_4 x - \mu h .$$

Because of (4.5), (4.7) and (1.7), and with $\mu \in (0, 1)$, it is clear that

$$|Q| \leq D_2 |\ddot{x}| + D_3 .$$

Thus if we multiply both sides of (4.8) by $\ddot{x}$ and integrate we have, x being ω-periodic, that

$$\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \leq |a_1^{-1}| \left(D_2 \int_{\tau}^{\tau+\omega} |\ddot{x}| \, |\ddot{x}| \, dt + D_3 \int_{\tau}^{\tau+\omega} |\ddot{x}| \, dt \right)$$

$$\leq D_4 \left(\int_{\tau}^{\tau+\omega} \ddot{x}^2 \, dt \right)^{\frac{1}{2}} \left(\int_{\tau}^{\tau+\omega} \ddot{x}^2 \, dt \right)^{\frac{1}{2}} + D_5 \left(\int_{\tau}^{\tau+\omega} \ddot{x}^2 \, dt \right)^{\frac{1}{2}}$$

by Schwarz's inequality. Hence, by (4.3),

$$\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \leq D_6 \left(\int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \right)^{\frac{1}{2}}$$

which shows at once that

$$(4.9) \qquad \int_{\tau}^{\tau+\omega} \dddot{x}^2 \, dt \leq D_7$$

for some D_7. Thus since $\ddot{x}(\tau) = 0$ for some τ we have from the identity:

$$\ddot{x}(t) = \ddot{x}(\tau) + \int_{\tau}^{t} \dddot{x}(s) \, ds$$

that

$$(4.10) \qquad \max |\ddot{x}(t)| \leq \int_{\tau}^{\tau+\omega} |\dddot{x}(s)| \, ds$$

$$\leq \omega^{\frac{1}{2}} \left(\int_{\tau}^{\tau+\omega} \dddot{x}^2(s) \, ds \right)^{\frac{1}{2}},$$

$$\leq D_8$$

by (4.9).

The estimates (4.5), (4.7) and (4.10) establish (2.9) and Theorem 1 then follows as was pointed out in § 2.

5. PROOF OF THEOREM 2

The procedure is exactly as in § 4 except that, in order to utilize the hypothesis (1.10) it is useful to note that

$$\int \gamma(x)\,\dot{x}^2\,\mathrm{d}t = \dot{x}\Gamma(x) - \int \ddot{x}\Gamma(x)\,\mathrm{d}t\,,$$

so that (4.2), in view of the ω-periodicity of x, also implies that

$$\int_{\tau}^{\tau+\omega} \ddot{x}^2\,\mathrm{d}t + \mu a^{-1}\int_{\tau}^{\tau+\omega} \ddot{x}\Gamma(x)\,\mathrm{d}t = \mu a^{-1}\int_{\tau}^{\tau+\omega} (h-p)\,\dot{x}\,\mathrm{d}t\,.$$

By (1.10) this leads in turn to the result:

$$\int_{\tau}^{\tau+\omega} \ddot{x}^2\,\mathrm{d}t \leq \mathrm{D}\left(\int_{\tau}^{\tau+\omega} |\ddot{x}|\,\mathrm{d}t + \int_{\tau}^{\tau+\omega} |\dot{x}|\,\mathrm{d}t\right), \tag{5.1}$$

$$\leq \mathrm{D}\left(\int_{\tau}^{\tau+\omega} \ddot{x}^2\,\mathrm{d}t\right)^{\frac{1}{2}},$$

by (4.1) and Schwarz's inequality, thus bringing us to the stage (4.3) of Theorem 1, from which point the rest of the proof of Theorem 2 can now follow exactly as in § 4.

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