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A n oscillatory and asymptotic classification of the solutions of differential equations with deviating arguments

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Equazioni differenziali ordinarie. — *An oscillatory and asymptotic classification of the solutions of differential equations with deviating arguments* (*). Nota di CHRISTOS G. PHILOS, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota, estendendo alcuni risultati ottenuti recentemente da Staikos e da Sficas, si classificano le soluzioni di una classe di equazioni differenziali ordinarie con argomenti deviati, rispetto al loro carattere oscillatorio e al loro comportamento per $t \to \infty$.

Let r_i $(i = 0, 1, \cdots, n)$ be positive continuous functions on the interval $[t_0, \infty)$. For a real-valued function h on $[T, \infty)$, $T \geqq t_0$, and any $\mu = 0, 1, \cdots, n$ we define the μ-th *r-derivative* of h by the formula

$$D_r^{(\mu)} h = r_\mu (r_{\mu-1} (\cdots (r_1 (r_0 h)')' \cdots)')' ,$$

when obviously we have

$$D_r^{(0)} h = r_0 h \qquad \text{and} \qquad D_r^{(i)} h = r_i (D_r^{(i-1)} h)' \qquad (i = 1, 2, \cdots, n).$$

Moreover, if $D_r^{(n)} h$ is defined on $[T, \infty)$, then h is said to be *n-times r-differentiable*. We note that in the case where $r_0 = r_1 = \cdots = r_n = 1$ the above notion of r-differentiability specializes to the usual one.

Now, we consider the n-th order $(n > 1)$ differential equation with deviating arguments of the form

$$(\mathrm{E}, \delta) \qquad (D_r^{(n)} x)(t) + \delta F(t; x[g_1(t)], x[g_2(t)], \cdots, x[g_m(t)]) = 0 ,$$

where $r_n = 1$ and $\delta = \pm 1$. The continuity of the real-valued functions F on $[t_0, \infty) \times \mathbf{R}^m$ and g_j $(j = 1, 2, \cdots, m)$ on $[t_0, \infty)$ as well as sufficient smoothness to guarantee the existence of solutions of (E, δ) on an infinite subinterval of $[t_0, \infty)$ will be assumed without mention. In what follows the term "solution" is always used only for such solutions $x(t)$ of (E, δ) which are defined for all large t. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function which is defined on an interval of the form $[T, \infty)$ is called *oscillatory* if it has no last zero, and otherwise it is called *nonoscillatory*.

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Furthermore, conditions (i) and (ii) below are assumed to hold throughout the paper:

(i) *For every* $j = 1, 2, \cdots, m$

$$\lim_{t \to \infty} g_j(t) = \infty .$$

(ii) *For every* $t \geqq t_0$,

$$F(t; 0, 0, \cdots, 0) = 0$$

and, moreover, $F(t; y)$ *is nondecreasing with respect to* y *in* $\mathbf{R}^m$.

Note. The order in $\mathbf{R}^m$ is considered in the usual sense, i.e.

$$y \leqq z \Longleftrightarrow (\forall j = 1, 2, \cdots, m)\, y_j \leqq z_j .$$

In this paper we shall classify all solutions of the differential equation (E, δ) with respect to their oscillatory character and to their behaviour at ∞. For this purpose, $S(\delta)$ will denote the set of all solutions of the equation (E, δ) and $S^{\sim}(\delta)$, $S^0(\delta)$, $S_1^{+\infty}(\delta)$, $S_2^{+\infty}(\delta)$, $S_1^{-\infty}(\delta)$, $S_2^{-\infty}(\delta)$, $S^{+\infty}(\delta)$, $S^{-\infty}(\delta)$ subsets of $S(\delta)$ defined as follows:

(*a*) $S^{\sim}(\delta)$ is the set of all oscillatory $x \in S(\delta)$.

(*b*) $S^0(\delta)$ is the set of all nonoscillatory $x \in S(\delta)$ with

$$\lim_{t \to \infty} (D_r^{(i)} x)(t) = 0 \qquad \text{monotonically} \quad (i = 0, 1, \cdots, n-1).$$

(*c*) $S_1^{+\infty}(\delta)$ is the set of all $x \in S(\delta)$ for which there exists an integer k, $0 \leqq k \leqq n-1$, with $n+k$ odd and such that:

(P_1) $\lim_{t \to \infty} (D_r^{(i)} x)(t) = \infty$ for every $i = 0, 1, \cdots, k$.

(P_2) If $k \leqq n-2$, then $\lim_{t \to \infty} (D_r^{(k+1)} x)(t)$ exists in $\mathbf{R}$.

(P_3) If $k \leqq n-3$, then for every $i = k+2, \cdots, n-1$

$$\lim_{t \to \infty} (D_r^{(i)} x)(t) = 0 ,$$

$$(D_r^{(i)} x)(t) \neq 0 \qquad \text{for all large } t,$$

$$(D_r^{(i)} x)(t)\, (D_r^{(i+1)} x)(t) \leqq 0 \qquad \text{for all large } t.$$

(*d*) $S_2^{+\infty}(\delta)$ is the set of all $x \in S(\delta)$ which possess properties (P_1)-(P_3) for some integer k, $0 \leqq k \leqq n-1$, with $n+k$ even.

(*e*) $S_1^{-\infty}(\delta)$ is the set of all $x \in S(\delta)$ for which the function $-x$ possesses properties (P_1)-(P_3) for some integer k, $0 \leqq k \leqq n-1$, with $n+k$ odd.

(*f*) $S_2^{-\infty}(\delta)$ is the set of all $x \in S(\delta)$ for which the function $-x$ possesses properties (P_1)-(P_3) for some integer k, $0 \leqq k \leqq n-1$, with $n+k$ even.

(*g*) $S^{+\infty}(\delta) = S_1^{+\infty}(\delta) \cup S_2^{+\infty}(\delta)$.

(*h*) $S^{-\infty}(\delta) = S_1^{-\infty}(\delta) \cup S_2^{-\infty}(\delta)$.

We introduce, now, the main conditions which will be used in the classification of the solutions of the equation (E , δ).

(C_1) *For every* $i = 1, 2, \cdots, n-1$

$$\int^{\infty} \frac{dt}{r_i(t)} = \infty .$$

(C_2) *For every nonzero constant* c *there exists an integer* λ, $0 \leqq \lambda \leqq \leqq n-1$, *such that*

$$\begin{cases} \displaystyle\int^{\infty} \left| F\left(t; \frac{c}{r_0[g_1(t)]}, \frac{c}{r_0[g_2(t)]}, \cdots, \frac{c}{r_0[g_m(t)]}\right) \right| dt = \infty , & \text{if } \lambda = n-1 \\ \displaystyle\int^{\infty} \frac{1}{r_{\lambda+1}(s_{\lambda+1})} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} \left| F\left(s; \frac{c}{r_0[g_1(s)]}, \frac{c}{r_0[g_2(s)]}, \cdots, \frac{c}{r_0[g_m(s)]}\right) \right| ds\, ds_{n-1} \cdots ds_{\lambda+1} = \infty , & \text{if } \lambda < n-1 . \end{cases}$$

(C_3) *For every nonzero constant* c,

$$\int^{\infty} \left| F\left(t; \frac{c}{r_0[g_1(t)]} \int_{t_0}^{g_1(t)} \frac{ds}{r_1(s)}, \frac{c}{r_0[g_2(t)]} \int_{t_0}^{g_2(t)} \frac{ds}{r_1(s)}, \cdots, \frac{c}{r_0[g_m(t)]} \int_{t_0}^{g_m(t)} \frac{ds}{r_1(s)}\right) \right| dt = \infty .$$

The oscillatory and asymptotic behavior of the solutions x of the differential equation (E , δ) with $x(t) = O(1/r_0(t))$ as $t \to \infty$ is well described by the following theorem due to the Author [1].

THEOREM 0. *Consider the differential equation* (E , δ) *subject to the conditions* (i), (ii), (C_1) *and* (C_2). *Then every solution* x *of the equation* (E , + 1) [*respectively*, (E , − 1)] *with* $x(t) = O(1/r_0(t))$ *as* $t \to \infty$ *for* n *even* [*resp. odd*] *is oscillatory, while for* n *odd* [*resp. even*] *is either oscillatory or such that*

$$\lim_{t\to\infty} (D_r^{(i)} x)(t) = 0 \qquad \textit{monotonically} \quad (i = 0, 1, \cdots, n-1).$$

In order to obtain our first result (Theorem 1) we need the following elementary lemma which has been proved by the Author in [1].

LEMMA. *Let* h *be an* n*-times* r*-differentiable function on* $[T, \infty)$, $T \geqq t_0$, *such that* $D_r^{(n)} h$ *is of constant sign on* $[T, \infty)$. *Moreover, let* μ, $0 \leqq \mu \leqq n-2$, *be an integer so that*

$$\int^{\infty} \frac{dt}{r_{\mu+1}(t)} = \infty .$$

If $\lim_{t\to\infty} (D_r^{(\mu)} h)(t)$ *is finite, then*

$$\lim_{t\to\infty} (D_r^{(\mu+1)} h)(t) = 0 .$$

THEOREM 1. *Consider the differential equation* (E, δ) *subject to the conditions* (i), (ii), (C_1) *and* (C_2). *Then for* n *even* [*resp. odd*] *the solutions of the equation* $(E, +1)$ [*resp.* $(E, -1)$] *admit the decomposition*

$$S(+1) = S^{\sim}(+1) \cup S^{+\infty}(+1) \cup S^{-\infty}(+1)$$

$$[\textit{resp.}\ S(-1) = S^{\sim}(-1) \cup S^{0}(-1) \cup S^{+\infty}(-1) \cup S^{-\infty}(-1)] ,$$

while for n *odd* [*resp. even*], *the decomposition*

$$S(+1) = S^{\sim}(+1) \cup S^{0}(+1) \cup S^{+\infty}(+1) \cup S^{-\infty}(+1)$$

$$[\textit{resp.}\ S(-1) = S^{\sim}(-1) \cup S^{+\infty}(-1) \cup S^{-\infty}(-1)] .$$

Proof. Let x be a nonoscillatory solution on an interval $[T_0, \infty)$, $T_0 \geqq t_0$, of the equation (E, δ) with $\limsup_{t\to\infty} |(D_r^{(0)} x)(t)| = \infty$. Without loss of generality, we suppose that $x(t) \neq 0$ for all $t \geqq T_0$. Next, by (i), we choose a $T \geqq T_0$ so that

$$g_j(t) \geqq T_0 \qquad \text{for every} \quad t \geqq T \qquad (j = 1, 2, \cdots, m).$$

Then, in view of (ii), equation (E, δ) yields

$$-\delta x(t)(D_r^{(n)} x)(t) = x(t) F(t; x[g_1(t)], x[g_2(t)], \cdots, x[g_m(t)]) \geqq$$
$$\geqq x(t) F(t; 0, 0, \cdots, 0) = 0$$

for every $t \geqq T$. Thus $D_r^{(n)} x$ is of constant sign on $[T, \infty)$ and so the functions $D_r^{(i)} x$ $(i = 1, 2, \cdots, n-1)$ are also eventually of constant sign.

Now, we consider the following two possible cases:

Case 1. $$\lim_{t\to\infty} (D_r^{(0)} x)(t) = \infty .$$

Let k be the greatest integer with $0 \leqq k \leqq n-1$ and

$$\lim_{t\to\infty} (D_r^{(i)} x)(t) = \infty \qquad \text{for every} \quad i = 0, 1, \cdots, k.$$

Obviously, if $k \leqq n-2$, then

$$\lim_{t\to\infty} (D_r^{(k+1)} x)(t) \qquad \text{exists in } \mathbf{R} .$$

So, if $k \leqq n-3$, then, by Lemma, for every $i = k+2, \cdots, n-1$

$$\lim_{t\to\infty} (D_r^{(i)} x)(t) = 0$$

and consequently it is easy to see that

$$(D_r^{(i)} x)(t)(D_r^{(i+1)} x)(t) \leqq 0 \qquad \text{for all large } t.$$

Finally, to derive that for $i = k+2, \cdots, n-1$.

$$(D_r^{(i)} x)(t) \neq 0 \qquad \text{for all large } t$$

it is enough to verify that $(D_r^{(n)} x)(t)$ is not identically zero for all large t. To do this, we suppose that there exists a $T_1 \geqq T$ such that

$$(D_r^{(n)} x)(t) = 0 \qquad \text{for every} \quad t \geqq T_1$$

and we consider a positive constant c so that for every $t \geqq T_0$

$$(D_r^{(0)} x)(t) \geqq c, \qquad \text{i.e.} \quad x(t) \geqq \frac{c}{r_0(t)}.$$

Then, in view of (ii), from equation (E, δ) we obtain

$$\begin{aligned} 0 = -\delta (D_r^{(n)} x)(t) &= F(t; x[g_1(t)], x[g_2(t)], \cdots, x[g_m(t)]) \\ &\geqq F\left(t; \frac{c}{r_0[g_1(t)]}, \frac{c}{r_0[g_2(t)]}, \cdots, \frac{c}{r_0[g_m(t)]}\right) \\ &\geqq F(t; 0, 0, \cdots, 0) = 0 \end{aligned}$$

for all $t \geqq T_1$. Therefore,

$$F\left(t; \frac{c}{r_0[g_1(t)]}, \frac{c}{r_0[g_2(t)]}, \cdots, \frac{c}{r_0[g_m(t)]}\right) = 0 \qquad \text{for every} \quad t \geqq T_1,$$

which contradicts (C_2).

Thus, x possesses properties (P_1)-(P_3), i.e. $x \in S^{+\infty}(\delta)$.

Case 2. $$\lim_{t\to\infty} (D_r^{(0)} x)(t) = -\infty.$$

Let k be the greatest integer with $0 \leqq k \leqq n-1$ and

$$\lim_{t\to\infty} (D_r^{(i)} x)(t) = -\infty \qquad \text{for every} \quad i = 0, 1, \cdots, k.$$

An argument similar to that used in Case 1 proves that the function $-x$ possesses properties (P_1)-(P_3), which means that $x \in S^{-\infty}(\delta)$.

We have proved that, if $\hat{S}(\delta)$ is the set of all solutions x of the equation (E, δ) with $\limsup_{t\to\infty} |(D_r^{(0)} x)(t)| = \infty$,

$$\hat{S}(\delta) = S^{\sim}(\delta) \cup S^{+\infty}(\delta) \cup S^{-\infty}(\delta).$$

This proves the theorem, since, by Theorem 0,

$$\overline{S}(+1) = S^{\sim}(+1) \text{ and } \overline{S}(-1) = S^{\sim}(-1) \cup S^0(-1), \qquad \text{if } n \text{ is even,}$$

$$\overline{S}(+1) = S^{\sim}(+1) \cup S^0(+1) \text{ and } \overline{S}(-1) = S^{\sim}(-1), \qquad \text{if } n \text{ is odd,}$$

where $\overline{S}(\delta)$ is the set of all solutions x of the equation (E, δ) with $x(t) = O(1/r_0(t))$ as $t \to \infty$.

THEOREM 2. *Consider the differential equation* (E, + 1) *subject to the conditions* (i), (ii), (C_1), (C_2) *and* (C_3). *Then for n even the solutions of the equation* (E, + 1) *admit the decomposition*

$$S(+1) = S^{\sim}(+1) \cup S_2^{+\infty}(+1) \cup S_2^{-\infty}(+1),$$

while for n odd, the decomposition

$$S(+1) = S^{\sim}(+1) \cup S^0(+1).$$

Proof. Let $x \in S^{+\infty}(+1)$ and k be the associated integer. The function x is a solution on an interval $[T_0, \infty)$, $T_0 > t_0$, of the equation (E, + 1). By property (P_1), x is eventually positive. Without loss of generality, we assume that $x(t) > 0$ for every $t \geqq T_0$.

Now, we suppose that $k \geqq 1$. Then, by property (P_1), we have

$$\lim_{t\to\infty} (D_r^{(0)} x)(t) = \lim_{t\to\infty} (D_r^{(1)} x)(t) = \infty$$

and consequently, using the Hospital rule, we can derive that

$$\lim_{t\to\infty} \frac{(D_r^{(0)} x)(t)}{\displaystyle\int_{t_0}^{t} \frac{ds}{r_1(s)}} = \infty .$$

So, there exists a positive constant c such that for every $t \geqq T_0$

$$(D_r^{(0)} x)(t) \geqq c \int_{t_0}^{t} \frac{ds}{r_1(s)}, \quad \text{i.e.} \quad x(t) \geqq \frac{c}{r_0(t)} \int_{t_0}^{t} \frac{ds}{r_1(s)} .$$

Thus, if, by (i), $T \geqq T_0$ is chosen so that

$$g_j(t) \geqq T_0 \qquad \text{for every} \quad t \geqq T \qquad (j = 1, 2, \cdots, m),$$

then, in view of (ii), from equation (E, + 1) we obtain

$$-(D_r^{(n-1)} x)(t) + (D_r^{(n-1)} x)(T) =$$

$$= \int_T^t F(s; x[g_1(s)], x[g_2(s)], \cdots, x[g_m(s)]) \, ds$$

$$\geqq \int_T^t F\left(s; \frac{c}{r_0[g_1(s)]} \int_{t_0}^{g_1(s)} \frac{du}{r_1(u)}, \frac{c}{r_0[g_2(s)]} \int_{t_0}^{g_2(s)} \frac{du}{r_1(u)}, \cdots \right.$$

$$\left. \cdots, \frac{c}{r_0[g_m(s)]} \int_{t_0}^{g_m(s)} \frac{du}{r_1(u)}\right) ds$$

for all $t \geqq T$. This, because of condition (C_3), gives

$$\lim_{t\to\infty} (D_r^{(n-1)} x)(t) = -\infty ,$$

a contradiction.

Hence, k must be zero, which obviously means that

$$x \in S_2^{+\infty}(+1), \qquad \text{if } n \text{ is even,}$$

$$x \in S_1^{+\infty}(+1), \qquad \text{if } n \text{ is odd.}$$

Next, we consider the case where n is odd. By (ii), equation $(E, +1)$ yields

$$(D_r^{(n)} x)(t) = -F(t; x[g_1(t)], x[g_2(t)], \cdots, x[g_m(t)]) \leqq$$
$$\leqq -F(t; 0, 0, \cdots, 0) = 0$$

for every $t \geqq T$, where $T, T \geqq T_0$, is chosen as above. From this and the property (P_3) it follows that

$$(D_r^{(2)} x)(t) > 0 \qquad \text{for every} \quad t \geqq T_1 ,$$

where $T_1, T_1 \geqq T$, can be chosen so that $(D_r^{(1)} x)(T_1) > 0$. Therefore, for every $t \geqq T_1$

$$(D_r^{(0)} x)(t) = (D_r^{(0)} x)(T_1) + \int_{T_1}^{t} \frac{1}{r_1(s)} (D_r^{(1)} x)(s)\, ds \geqq (D_r^{(1)} x)(T_1) \int_{T_1}^{t} \frac{ds}{r_1(s)}$$

and so it is easy to see that there exists a positive constant K such that

$$(D_r^{(0)} x)(t) \geqq K \int_{t_0}^{t} \frac{ds}{r_1(s)} \qquad \text{for all} \quad t \geqq T_0 .$$

Thus, the contradiction $\lim_{t\to\infty} (D_r^{(n-1)} x)(t) = -\infty$ can again be derived in the considered case of odd n.

We have proved that

$$S^{+\infty}(+1) = S_2^{+\infty}(+1), \qquad \text{if } n \text{ is even,}$$

$$S^{+\infty}(+1) = \emptyset , \qquad \text{if } n \text{ is odd.}$$

A similar argument gives

$$S^{-\infty}(+1) = S_2^{-\infty}(+1), \qquad \text{if } n \text{ is even,}$$

$$S^{-\infty}(+1) = \emptyset , \qquad \text{if } n \text{ is odd}$$

and hence Theorem 1 completes the proof of our theorem.

THEOREM 3. *Consider the differential equation* $(E, -1)$ *subject to the conditions* (i), (ii), (C_1), (C_2) *and* (C_3). *Then for n even the solutions of the equa-*

tion (E , — 1) *admit the decomposition*

$$S(-1) = S^{\sim}(-1) \cup S^{0}(-1) \cup S_{1}^{+\infty}(-1) \dot{\cup} S_{1}^{-\infty}(-1),$$

while for n odd, the decomposition

$$S(-1) = S^{\sim}(-1) \cup S_{1}^{+\infty}(-1) \cup S_{1}^{-\infty}(-1).$$

Proof. We suppose that $S_2^{+\infty}(-1) \neq \varnothing$ and we consider a solution $x \in S_2^{+\infty}(-1)$ as well as the associated integer k. The function x is a solution on an interval $[T_0, \infty)$, $T_0 > t_0$, of $(E, -1)$. Because of (P_1), we have $x(t) > 0$ for all large t. Without loss of generality, we assume that x is positive on the whole interval $[T_0, \infty)$.

Suppose that $k \geqq 1$. Then, as in the proof of Theorem 2, we conclude that there exists a positive constant c such that

$$(D_r^{(0)} x)(t) \geqq c \int_{t_0}^{t} \frac{ds}{r_1(s)} \qquad \text{for every} \quad t \geqq T_0 .$$

So, by (ii), from equation (E , — 1) we obtain

$$(D_r^{(n-1)} x)(t) - (D_r^{(n-1)} x)(T) =$$

$$= \int_{T}^{t} F(s; x[g_1(s)], x[g_2(s)], \cdots, x[g_m(s)])\, ds$$

$$\geqq \int_{T}^{t} F\left(s; \frac{c}{r_0[g_1(s)]} \int_{t_0}^{g_1(s)} \frac{du}{r_1(u)}, \frac{c}{r_0[g_2(s)]} \int_{t_0}^{g_2(s)} \frac{du}{r_1(u)}, \cdots \right.$$

$$\left. \cdots, \frac{c}{r_0[g_m(s)]} \int_{t_0}^{g_m(s)} \frac{du}{r_1(u)}\right) ds$$

for all $t \geqq T$, where T, $T \geqq T_0$, is chosen, by (i), so that

$$g_j(t) \geqq T_0 \qquad \text{for every} \quad t \geqq T \qquad (j = 1, 2, \cdots, m).$$

This, because of condition (C_3), gives

$$\lim_{t \to \infty} (D_r^{(n-1)} x)(t) = \infty .$$

But $k \leqq n - 2$, since $n + k$ is even, and consequently the last relation is a contradiction.

Thus, k must be zero and therefore n is even. Furthermore, in view of (ii), from equation (E , — 1) we have

$$(D_r^{(n)} x)(t) = F(t; x[g_1(t)], x[g_2(t)], \cdots, x[g_m(t)]) \geqq$$

$$\geqq F(t; 0, 0, \cdots 0) = 0, \ t \geqq T .$$

Namely, $D_r^{(n)} x$ is nonnegative on $[T, \infty)$. By this and (P_3), there exists a $T_1 \geqq T$ such that $(D_r^{(1)} x)(T_1) > 0$ and

$$(D_r^{(2)} x)(t) \geqq 0 \qquad \text{for every } t \geqq T_1 .$$

Hence, as in the proof of Theorem 2, we conclude the existence of a constant $K > 0$ so that

$$(D_r^{(0)} x)(t) \geqq K \int_{t_0}^{t} \frac{ds}{r_1(s)} \qquad \text{for all } t \geqq T_0$$

ans so the contradiction $\lim_{t \to \infty} (D_r^{(n-1)} x)(t) = \infty$ can again be derived.

We have therefore proved that $S_2^{+\infty}(-1) = \varnothing$. By a similar argument, we obtain $S_2^{-\infty}(-1) = \varnothing$. Finally, Theorem 1 completes the proof of our theorem.

Remark. In the usual case where $r_0 = r_1 = \cdots = r_{n-1} = 1$, the condition (C_1) holds by itself while the condition (C_2) becomes (cf. [1]):

(C_2^0) *For every nonzero constant* c,

$$\int^{\infty} t^{n-1} \left| F(t; c, c, \cdots, c) \right| dt = \infty .$$

Moreover, the condition (C_3) takes the form:

(C_3^0) *For every nonzero constant* c,

$$\int^{\infty} \left| F(t; cg_1(t), cg_2(t), \cdots, cg_m(t)) \right| dt = \infty .$$

So, by applying our results for the differential equation

$$x^{(n)}(t) + \delta F(t; x[g_1(t)], x[g_2(t)], \cdots, x[g_m(t)]) = 0 ,$$

we obtain recent ones due to Staikos and Sficas [2].

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