ATTI ACCADEMIA NAZIONALE DEI LINCEI

CLASSE SCIENZE FISICHE MATEMATICHE NATURALI

RENDICONTI

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On the Paley-Wiener-Schwartz theorem in infinite dimensions

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **63** (1977), n.3-4, p. 192–194.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1977_8_63_3-4_192_0>

Analisi funzionale. — *On the Paley-Wiener-Schwartz theorem in infinite dimensions.* Nota (*) di TEÒFILO ABUABARA, presentata dal Corrisp. G. CIMMINO.

RIASSUNTO. — Si risolve una questione posta da L. Nachbin, dando condizioni caratterizzanti le trasformate di Fourier delle distribuzioni a supporto limitato su certi spazi di Banach a infinite dimensioni.

This Note concerns a generalization of the Paley-Wiener-Schwartz theorem for distributions on an infinite dimensional Banach space. Nachbin and Dineen [1] defined the Fréchet space $\varepsilon_{Nbc}(E ; F)$ of all infinitely nuclearly differentiable mappings of bounded-compact type from E to F, with E a real Banach space and F any Banach space. This is the closure in $\varepsilon_{Nb}(E ; F)$ of the subspace generated by the mappings of the form $\Phi^m \cdot b : E \to F , \Phi \in E'$, $b \in F , m \in N$. We recall that $\varepsilon_{Nb}(E ; F)$ is the Fréchet space of all infinitely differentiable mappings $f : E \to F$ such that $\hat{d}^m f(E) \subset \mathscr{P}_N({}^mE ; F)$ for $m = 0, 1, 2, \cdots$, each mapping $\hat{d}^m f : E \to \mathscr{P}_N({}^mE ; F)$ being differentiable of first order and bounded on bounded subsets. Here $\mathscr{P}_N({}^mE ; F)$ denotes the Banach space of nuclear m-homogeneous polynomials from E to F, endowed with the nuclear norm. For further details we rever to Nachbin [2]. The topology of $\varepsilon_{Nb}(E ; F)$ is the one generated by the following countable system of semi-norms $f \in \varepsilon_{Nb}(E ; F) \xmapsto{q_{m,n}} q_{m,n}(f) = \sup \{\| \hat{d}^k f(x) \|_N ; k \leq n$, $\| x \| \leq m\}$, for $m, n = 0, 1, 2, \cdots$. In the case that E is finite dimensional and $F = \mathbf{C}$, $\varepsilon_{Nbc}(E ; \mathbf{C}) = \varepsilon_{Nbc}(E)$ is simply the space $\varepsilon(E)$ endowed with the Schwartz topology [4]. For this reason and on account of Theorem 1 below, $\varepsilon'_{Nbc}(E)$, the dual space to $\varepsilon_{Nbc}(E)$, is called the space of distributions with bounded support in infinite dimensions. Nachbin and Dineen proved that the Paley-Wiener-Schwartz conditions are not sufficient if E is infinite dimensional, by constructing a holomorphic function of exponential type on $(E')_{\mathbf{C}}$ (the normed complexification of E′) bounded on E′ (and hence slowly increasing) which is not the Fourier transform of any element of $\varepsilon'_{Nbc}(E)$. The main result of the present article is a necessary and sufficient condition for a slowly increasing complex valued holomorphic function of exponential type on $(E')_{\mathbf{C}}$ (where E belongs to a wide class of separable Banach spaces) to be the Fourier transform of a distribution with bounded support in infinite dimensions: the

(*) Pervenuta all'Accademia il 10 ottobre 1977.

(1) This is an announcement of the main results in the Author's doctoral thesis at IMPA, Rio de Janeiro (1977), written under the guidance of Prof. Jaime Lesmes.

Paley-Wiener-Schwartz theorem in infinite dimensions. Following Restrepo [3], we say that a Banach space E has Property (B) if there exists a sequence $P_n : E \to E$ of continuous linear projections such that each $P_n(E)$ is finite dimensional, $P_n(x) \to x$ for every $x \in E$ and $P_n^*(\varphi) \to \varphi$ for every $\varphi \in E'$ where P_n^* denotes the adjoint operator of P_n. Every Banach space with a biorthogonal basis has Property (B). In particular, every Hilbert space has Property (B). We say that $\xi \in \varepsilon'_{Nbc}(E)$ has support contained in the closed ball in E of center O and radius m, if there exists constant $c > 0$ and $\nu \in \mathbf{N}$ such that

$$|\xi(g)| \leq c q_{m,\nu}(g)$$

for every $g \in \varepsilon_{Nbc}(E)$.

THEOREM 1. *Let* E *be a real separable Banach space with Property* (B) *and denote by* Y *the vector subspace of* $\varepsilon_{Nbc}(E)$ *generated by the mappings of the form* $e^{i\varphi} : E \to \mathbf{C}$, *where* $\varphi \in E'$.

a) *If* $\xi \in \varepsilon'_{Nbc}(E)$ *has support contained in the closed ball of center* O *and radius* m *and, if* $f : (E')_{\mathbf{C}} \to \mathbf{C}$ *is defined by* $f(\zeta) = \xi(e^{i\zeta})$ *then*: 1) f *is a holomorphic function on* $(E')_{\mathbf{C}}$ *and there exists constants* $c > 0$ *and* $\nu \in N$ *such that*

$$|f(\zeta) \leq c(1 + \|\zeta\|)^{\nu} \exp(m \|\operatorname{Im} \zeta\|)$$

for every $\zeta \in (E')_{\mathbf{C}}$, *where* Im ζ *denotes the imaginary part of* ζ. 2) *The sequence* $(\xi_n)_n \subset Y'$, *where* ξ_n *is defined by*

$$g = \sum_{j=1}^{l} \alpha_j e^{i\varphi_j} \in Y \mapsto \xi_n(g) = \sum_{j=1}^{l} \alpha_j f(\varphi_j \circ P_n),$$

is equicontinuous.

b) *Conversely, if* $f : (E')_{\mathbf{C}} \to \mathbf{C}$ *is a function satisfying* 1) *and* 2), *then there exists* $\xi \in \varepsilon'_{Nbc}(E)$ *with support contained in the closed ball in* E *of center* O *and radius* αm *such that* $\xi(e^{i\varphi}) = \hat{\xi}(\varphi) = f(\varphi)$ *for every* $\varphi \in E'$, *where* α *is a constant such that* $\sup_n \|P_n\| \leq \alpha$.

In the following, we sketch the proof of the theorem. In the direct part, f is a holomorphic function of exponential type on $(E')_{\mathbf{C}}$ and slowly increasing on E'. We have that $\xi_n(g) = \xi(g \circ P_n)$ and $q_{m,k}(g \circ P_n) \leq \alpha^k q_{m,k}(g)$, for every $m, k \in \mathbf{N}$ and for every $g \in Y$. Hence the equicontinuity of the sequence $(\xi_n)_n$ follows from the continuity of ξ. To prove the converse of the theorem, one passes to the finite dimensional case to apply the Paley-Wiener-Schwartz theorem, and then uses the Alaoglu-Bourbaki theorem to show existence of ξ. The assertion on the support follows from the Paley-Wiener-Schwartz theorem and the Mackey theorem.

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