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Age-dependent population dynamics


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Analisi matematica. — Age-dependent population dynamics.
Nota (*) di Gabriella Di Blasio (**) e Lamberto Lambert (***),
presentata dal Corrisp. G. Stampacchia.

RIASSUNTO. — Si studia il problema di Cauchy per una equazione differenziale deri-
vante dallo studio della diffusione di una singola specie biologica. Si dimostra l’esistenza e
l’unicità della soluzione di tale problema e la dipendenza continua dai dati.

1. INTRODUCTION

This paper is concerned with the study of the following partial differential
equation
\[ \frac{\partial}{\partial t} u(t, a, x) + \frac{\partial}{\partial a} u(t, a, x) = -\mu(a) u + \int_{0}^{+\infty} k(a, a') \Delta u \, da' \]
\[ t, a \geq 0, \quad x \in \Omega \]
which has been proposed by Gurtin as a model for diffusion of a single species
population [4].

Here \( u(t, a, x) \) represents the density per unit volume and age of some
biological population at time \( t \) at the location \( x \) in \( \Omega \) and \( \mu \) is the age-dependent
mortality rate so that \( -\mu u \) represents the death process.

We shall study equation (1) together with the following additional con-
ditions

(i) an initial space and age distribution \( u(0, a, x) = u_0(a, x) \);
(ii) an age boundary condition representing the birth process
\( u(t, 0, x) = b(t, x) \);
(iii) a spatial boundary condition \( \frac{\partial}{\partial n} u = 0 \) where \( \frac{\partial}{\partial n} \) is the exterior
normal derivative at the boundary \( \partial \Omega \).

2. PRELIMINARIES

In this section we collect some known results concerning dissipative
functions (see [2] and [3]).

Let \( H \) be a real Hilbert space; a function \( A : D(A) \subseteq H \rightarrow H \) is said to
be dissipative if for each \( u, v \in D(A) \) we have \( \langle Au - Av, u - v \rangle \leq 0 \). A
dissipative \( A \) is said to be hyper-dissipative if \( \langle \lambda I - A \rangle (D(A)) = H \) for each
\( \lambda > 0 \).

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If \( A \) is hyper-dissipative we can consider the Yosida approximating functions \( A_n : H \to H, \quad (n \in \mathbb{N}) \)-defined by

\[
A_n = n (nI - A)^{-1} n - nI = A (nI - A)^{-1} n.
\]

The following Lemma collects some known properties of the functions \( A_n \).

**Lemma 1.** If \( A \) is hyper-dissipative then:

(i) \( A_n \) is a Lipschitz continuous function;

(ii) \( A_n \) is hyper-dissipative;

(iii) if \( u_n \) is such that \( u_n \to u \) and \( A_n u_n \rightharpoonup v \) (weak convergence) then \( u \in D(A) \) and \( Au = v \).

3. **Properties of the functions** \( \frac{\partial}{\partial a} + \mu I \) and \( \int k \Delta \)

Let \( \mu : [0, +\infty) \to \mathbb{R}, \ a \to \mu (a) \geq 0 \) and \( K : [0, +\infty] \times [0, +\infty] \to \mathbb{R}, (a', a) \to K (a, a') \) be measurable functions; we shall study equation (i) under the following hypotheses

\( (m_1) \quad \mu \in L^1_{\text{loc}} ([0, +\infty]) \)

\( (k_1) \) there exists a constant \( c_1 \) such that for each \( u \in L^2 ([0, +\infty]) \) we have

\[
\int_0^{+\infty} \left( \int_0^{+\infty} K (a, a') u (a') \, da' \right)^2 \, da \leq c_1 \left( \int_0^{+\infty} u^2 \, da \right)^{\frac{1}{2}}
\]

\( (k_2) \) for each \( u \in L^2 ([0, +\infty]) \) we have

\[
\int_0^{+\infty} \int_0^{+\infty} K (a, a') u (a) u (a') \, da' \, da \geq 0.
\]

Now let \( \Omega \subseteq \mathbb{R}^n \) be an open bounded set with smooth boundary \( \partial \Omega \) and set \( H = L^2 ([0, +\infty]) \); we denote by \( A_\beta \), \( T \) and \( B \) the operators defined by

\[
D(A_\beta) = \begin{cases} u \in H, \ a \to u (a, x) \in W^{1,2} ([0, +\infty]) & \text{and} \quad u (0, x) = \beta \\ \text{for a.e. } x \in \Omega; \quad (a, x) \to \frac{\partial}{\partial a} u (a, x) \in H \end{cases}
\]

\[
A_\beta u = - \frac{\partial}{\partial a} u
\]

where \( \beta \in L^2 (\Omega) \) is a given function and

\[
\begin{align*}
D(T) &= H \\
Tu &= \int_0^{+\infty} K (a, a') u (a', x) \, da'
\end{align*}
\]
The following theorems are well known

**Theorem 2.** $A_\beta - \mu I$ is hyper-dissipative and we have $(A_\beta u - \mu u, u) \leq \frac{1}{2} \int \beta^2 \, dx$.

**Theorem 3.** $B$ is hyper-dissipative.

The following Lemmas collect some further properties of $A_\beta, T$ and $B$.

**Lemma 2.** The operator $TB$ is dissipative.

**Proof.** Let $u \in D(B)$ from ($k_2$) we have

$$(TBu, u) = -\int_0^{+\infty} \int_0^{+\infty} \int K(a', x) \sum_{i=1}^m u_{x_i}(a', x) u_{x_i}(a, x) \, da' \, da \, dx \leq 0.$$
4. Existence and uniqueness results

For each \( s \geq 0 \) and \( n \in \mathbb{N} \) we shall consider the following regularized problem

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} &= -\mu u + \int_{0}^{+\infty} K(a, a') \Delta_n u \, da' + \varepsilon \Delta_n u + w \\
u(t, o, x) &= b(t, x) \\
u(o, a, x) &= \nu_0(a, x)
\end{align*}
\]

where \( \Delta_n u = B_n u(t) \) with \( B_n \) defined as in section 3.

To study problem (2) it is convenient to introduce the following definition.

We say that a function \( u \in C(o, T; H) \) is a strong solution of (2) if there exists \( \{u_k\} \) such that

\[(s_1) \quad u_k \in W^{1,2}(o, T; H) ; \quad \mu u_k \in L^2(o, T; H) ; \quad a \to u_k(t, a, x) \in \mathbb{R} \quad \text{for a.e.} \ (t, a) \in [o, T] \times \Omega \quad \text{and} \quad t \to \frac{\partial}{\partial a} \frac{\partial u_k}{\partial a} \in L^2(o, T; H)\]

\[(s_2) \quad \text{we have } u_k \to u \text{ in } C(o, T; H)\]

\[
\begin{align*}
\frac{\partial u_k}{\partial t} + \frac{\partial u_k}{\partial a} + \mu u_k &\to \int_{0}^{+\infty} K \Delta_n u \, da' + \varepsilon \Delta_n u + w \quad \text{in } L^2(o, T; H) \\
u_k(t, o, x) &\to b(t, x) \quad \text{in } C(o, T; L^2(\Omega)) \\
u_k(o, a, x) &\to \nu_0(a, x) \quad \text{in } H.
\end{align*}
\]

It is not difficult to prove the following result

**Lemma 6.** Let \( K = o, \varepsilon = o, w = o \) and \( \nu_0 = o \) then there exists a unique strong solution \( u_1 \) of (2) and we have

\[
u_1(t, a, x) = \begin{cases} b(t - a, x) \exp \left( \int_{0}^{a} \mu(\sigma) \, d\sigma \right) & t > a \\ 0 & t \leq a. \end{cases}
\]

(1) If \( E \) is a Hilbert space we denote by \( C(o, T; E) \) the Banach space of all continuous functions \( u : [o, T] \to E \); by \( L^2(o, T; E) \) the Hilbert space of square integrable functions \( u : [o, T] \to E \) and by \( W^{1,2}(o, T; E) \) the space of all absolutely continuous functions \( u : [o, T] \to E \) such that \( (d/dt) u \in L^2(o, T; E) \).
Lemma 7. Let $b = 0$ and $w = \int_0^{+\infty} K(a, a') \Delta_n u_1 \, da' + \varepsilon \Delta_n u_1$ then there exists a unique $\bar{u}_{e,n}$ strong solution of (2).

By Theorem 2 and Lemma 4 we have that $A_0 - \mu I$ is hyper-dissipative and that $TB_n + \varepsilon B_n$ is continuous and dissipative so that (see [1, Theorem 1]) $A_0 - \mu I + TB_n + \varepsilon B_n$ is hyper-dissipative and the result follows (see [2] and [3]).

Summarizing we have

Theorem 4. Let $u_1$ and $\bar{u}_{e,n}$ be the functions defined as in Lemmas 6, 7; then the function $u_{e,n} = u_1 + \bar{u}_{e,n}$ is a strong solution of (2) with $w = 0$.

The following Lemma collects some a-priori estimates for the solutions of (2).

Lemma 8. Let $u_0 \in D(B)$ and let $z_{e,n}$ be the strong solution of (2) given by Theorem 4. Then for each $t \in [0, T]$ we have:

(i) $\| u_{e,n}(t) \|_H \leq \| u_0 \|_H + \| b \|_{C(0,T;L^2(\Omega))}$

(ii) $\varepsilon \int_0^t \| \Delta_n u_{e,n} \|^2 \, ds \leq \tilde{C}(T, u_0, b)$

where $\tilde{C}(T, u_0, b)$ is a constant depending on $T, u_0$ and $b$.

Moreover if $u_0, \bar{u}_0 \in D(B), b, \bar{b} \in C(0, T; L^2(\Omega))$ and if $u, \bar{u}_{e,n}$ are the corresponding strong solutions then

(iii) $\| u_{e,n}(t) - \bar{u}_{e,n}(t) \|_H \leq \| u_0 - \bar{u}_0 \|_H + \| b - \bar{b} \|_{C(0,T;L^2(\Omega))}$.

Proof. To prove (i) it suffices to take the scalar product of $(e_2)$ with $u_k$, integrate over $[0,t]$ and use Theorem 2. Assertion (ii) follows from Lemma 5 by taking the scalar product of $(e_2)$ with $\Delta_n u_k$. Finally the proof of (iii) is similar to that of (i).

Finally using Lemma 8 (ii) and Lemma 1 (iii) we get the following existence result for the solutions of (2) in the generalized sense specified below

Theorem 5. Let $u_0 \in D(B)$ and $b \in C(0, T; L^2(\Omega))$ then there exists $u$ and $\{u_e\}$ verifying the following properties

$$(g_1) \quad x \rightarrow u_e(t, a, x) \in W^{2,2}(\Omega) \quad \text{for a.e.} \quad (t, a) \in [0, T] \times [0, +\infty[ \quad \text{and} \quad \frac{3}{2} \varepsilon u = 0 \quad \text{a.e.} \quad x \in \partial \Omega$$
(g\_f) \( u_n \) is the strong solution of the problem

\[
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \mu u = \int_0^{+\infty} K(a, a') \Delta u_n \, da' + \varphi_n
\]

\[
u(t, 0, x) = b(t, x)
\]

\[
u(0, a, x) = u_0(a, x)
\]

\( \gamma \) we have \( u_n \to u \) in \( C(0, T; H) \) and \( \varphi_n \to 0 \) in \( L^2(0, T; H) \).

Moreover if \( u_0, \tilde{u}_0 \in D(B) \), \( b, \tilde{b} \in C(0, T; L^2(\Omega)) \) and \( u, \tilde{u} \) are the corresponding generalized solutions then

\[
\| u(t) - \tilde{u}(t) \|_H \leq \| u_0 - \tilde{u}_0 \|_H + \| b - \tilde{b} \|_{C(0,T;L^2(\Omega))}.
\]

REFERENCES