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On $A(R, \lambda_n, k)$ summability methods

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Analisi matematica. — *On* $A(R, \lambda_n, k)$ *summability methods.* Nota(*)
di BABBAN PRASAD MISHRA e DINESH SINGH, presentata dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori introducono la sommazione $A(R, \lambda_n, k)$ e ne mettono in evidenza alcune proprietà.

INTRODUCTION

As it is familiar that Amir [1] defined the (A, k) summability method by considering the iteration product of two summability methods (A) and (C, k), it is natural to define a summability method $A(R, \lambda_n, k)$ by considering the product of (A) and (R, λ_n, k) methods. In this paper, we have introduced the summability method $A(R, \lambda_n, k)$ and investigated some of its properties.

1. Let $\{\lambda_n\}$ be an arbitrary sequence of numbers such that

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n .$$

We write, for $t > 0$,

$$A_\lambda(t) = \sum_{\lambda_v \leq t} a_v$$

and

$$A_\lambda^k(t) = \sum_{\lambda_v \leq t} (t - \lambda_v)^k a_v = \int_0^t (t - T)^k \, dA_\lambda(T) ,$$

where $k > 0$.

We define

$$A_\lambda^0(t) = A_\lambda(t) .$$

We say that the series $\sum_{n=0}^{\infty} a_n$ is summable

(*i*) (R, λ_n, k) to the sum s provided the function

$$C_\lambda^k(t) = t^{-k} A_\lambda^k(t)$$

tends to s as $t \to \infty$,

(*ii*) (A, λ_n) to the sum s provided

$$f(x) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n x} \tag{1.1}$$

is convergent for all positive x and tends to s as $x \to 0+$.

(*) Pervenuta all'Accademia il 26 luglio 1977.

It is familiar that the (C, k), $k > -1$, summability of a series implies the (A) summability to the same sum. The analogy breaks down in the case of Riesz and Abel typical methods except in the case $\lambda_n = n$. The additional condition that the series defining $f(x)$ is convergent for all $x > 0$ is required to overcome this problem. It can be readily shown that the convergence of (1.1) for $x > 0$ implies the absolute convergence of

$$(1.2) \qquad x \int_0^\infty A_\lambda(u)\, e^{-ux}\, du$$

and both tend to the same sum as $x \to 0+$, i.e. if we assume (1.2) as A (R, λ_n, 0) mean, then

$$(1.3) \qquad (A, \lambda_n) \subseteq A(R, \lambda_n, 0).$$

Suppose that, for $k > 0$,

$$(1.4) \qquad f_k(x) = x \int_0^\infty C_\lambda^k(u)\, e^{-ux}\, du.$$

It is easy to show that the convergence of (1.1) implies the absolute convergence of (1.4). We now define the A (R, λ_n, k) summability method as follows:

We say that the series $\sum_{n=0}^{\infty} a_n$ is summable A (R, λ_n, k) $k \geq 0$, to the sum s provided the series (1.1) is convergent for all $x > 0$ and $f_k(x)$ tends to a limit s as $x \to 0+$.

We shall use the following relations in the sequel.

If $k > 0$, $l > 0$, then

$$A_\lambda^{k+l}(x) = \frac{\Gamma(k+l+1)}{\Gamma(k+1)\Gamma(l)} \int_0^x (x-t)^{l-1} A_\lambda^k(t)\, dt.$$

For $k > 1$, we have

$$\frac{d}{dt}[A_\lambda^k(t)] = k A_\lambda^{k-1}(t)$$

and, for $k > l \geq 0$,

$$\frac{\Gamma(k-l)\Gamma(l+1)}{\Gamma(k+1)} = t^{l+1} \int_0^\infty (\rho + t)^{-k-1} \cdot \rho^{k-l-1}\, d\rho,$$

$$\Gamma(k) = t^k \int_0^\infty \rho^{k-1} e^{-\rho t}\, d\rho.$$

2. In this paper, we shall consider the proof of following theorems.

THEOREM 2.1. *The* $A(R, \lambda_n, k)$ *method is regular for* $k \geq 0$.

THEOREM 2.2. *If* $k > l \geq 0$, *then* $\sum_{n=0}^{\infty} a_n = sA(R, \lambda_n, k)$ *whenever* $\sum_{n=0}^{\infty} a_n = sA(R, \lambda_n, l)$.

THEOREM 2.3. *If the series* $\sum_{n=0}^{\infty} a_n$ *is summable* (R, λ_n, k) *to the sum* s *for some* $k \geq 0$, *then the series* $\sum_{n=0}^{\infty} a_n$ *is summable* $A(R, \lambda_n, l)$ *to the same sum for any* $l \geq 0$.

THEOREM 2.4. *If* $k \geq 1$, $k \geq l \geq 0$ *and the series* $\sum_{n=0}^{\infty} a_n$ *is summable* $A(R, \lambda_n, l)$ *to the sum* s, *then a necessary and sufficient condition that* $\sum_{n=0}^{\infty} a_n$ *is summable* $(R, \lambda_n, k-1)$ *to the sum* s, *is that*

$$\overline{C}_\lambda^k(u) = u^{-k}\, \overline{A}_\lambda^k(u) = u^{-k} \sum_{\lambda_n \leq u} (u - \lambda_n)^{k-1} \lambda_n a_n = o(1) \tag{2.1}$$

as $u \to \infty$.

3. The Proof of Theorem 2.1 follows immediately from the regularity of the (A, λ_n) method and Theorem 2.2, with $l = 0$ and k replaced by l.

To prove Theorem 2.2, we require the following lemma:

LEMMA 3.1. *If* $k > l \geq 0$ *and if the series* (1.1) *is convergent for all* $x > 0$, *then*

$$f_k(x) = \frac{\Gamma(k+1)}{\Gamma(k-l)\,\Gamma(l+1)}\, x^{l+1} \int_0^\infty u^{k-l-1} (u+x)^{-k-1} f_l(u+x)\, du \,. \tag{3.1}$$

Proof of the Lemma. The convergence of (1.1) for $x > 0$ justifies the inversions in the right hand side of (3.1). Introducing the value of $f_l(u+x)$ in (3.1) and inverting the order of integrations, the right hand side of (3.1) becomes

$$\frac{\Gamma(k+1)}{\Gamma(k-l)\,\Gamma(l+1)}\, x^{l+1} \int_0^\infty A_\lambda^l(p)\, p^{-l}\, dp \cdot \int_0^\infty u^{k-l-1} (u+x)^{-k} e^{-p(u+x)}\, du \,. \tag{3.2}$$

After the transformation

$$p(u+x) = (p+u')\, x$$

in u integral of (3.2), this becomes

$$\frac{\Gamma(k+1)}{\Gamma(k-l)\,\Gamma(l+1)}\, x \int_0^\infty A_\lambda^l(p)\, dp \int_0^\infty (u'+p)^{-k} \cdot u'^{k-l-1} e^{-(p+u')x}\, du'$$

$$= \frac{\Gamma(k+1)}{\Gamma(k-l)\,\Gamma(l+1)}\, x \int_0^\infty u'^{-k} e^{-u'x}\, du' \int_0^{u'} (u'-p)^{k-l-1} A_\lambda^l(p)\, dp$$

$$= x \int_0^\infty C_\lambda^k(u')\, e^{-u'x}\, du' = f_k(x)$$

and hence the lemma is completely established.

For the proof of Theorem 2.2, we observe that $F_{l'}(x)$ is convergent for all $x > 0$ and $l' \geq 0$ and with $l' = l$ tends to s as $x \to 0+$. Furthermore,

$$\frac{\Gamma(k+1)}{\Gamma(k-l)\,\Gamma(l+1)}\, x^{l+1} \int_0^\infty (u+x)^{-k-1}\, u^{k-l-1}\, du = 1\,.$$

It follows that

$$(3.3)\quad f_k(x) - s = \frac{\Gamma(k+1)}{\Gamma(k-l)\,\Gamma(l+1)}\, x^{l+1} \int_0^\infty (u+x)^{-k-1}\, u^{k-l-1}\, \{f_l(u+x) - s\}\, du.$$

But it can be easily shown that the expression on the right hand side of (3.3) tends to zero as $x \to 0+$. Thus, we get the desired result.

We now consider the proof of Theorem 2.3. It is known that if $\sum_{n=0}^{\infty} a_n$ is summable (R, λ_n, k) and if the (A, λ_n) method is applicable, then $\sum_{n=0}^{\infty} a_n$ is summable (A, λ_n) (1). Now Theorem 2.3 is an immediate consequence of Theorem 2.2. and (1.3).

We end this section with the proof of Theorem 2.4. The necessary part of the theorem is an immediate consequence of the known identity

$$(3.4)\qquad \bar{C}_\lambda^k(u) = C_\lambda^{k-1}(u) - C_\lambda^k(u)\,.$$

For the sufficiency part of the theorem, we observe that

$$(3.5)\qquad C_\lambda^k(u) = k\int_0^u \frac{\bar{C}_\lambda^k(p)}{p}\, dp\,,\ k > 0\,.$$

Furthermore,

$$(3.6)\qquad f_k(x) = kx\int_0^\infty e^{-px} dp \int_0^p \frac{\bar{C}_\lambda^k(u)}{u}\, du$$

$$= k\int_0^\infty p^{-k-1}\, e^{-px}\, \bar{A}_\lambda^k(p)\, dp = k\int_0^\infty p^{-1}\, e^{-px}\, \bar{C}_\lambda^k(p)\, dp\,.$$

It follows, from (3.5) and (3.6) that

$$C_\lambda^k(u) - f_k(x) = k\int_0^u \frac{\bar{C}_\lambda^k(p)}{p}\,(1 - e^{-px})\, dp$$

$$- k\int_u^\infty p^{-1}\, \bar{C}_\lambda^k(p)\, e^{-px}\, dp\,.$$

Now

$$k\int_0^u \frac{\bar{C}_\lambda^k(p)}{p}\,(1 - e^{-px})\, dp = O\left[x\int_0^u \bar{C}_\lambda^k(p)\, dp\right] = o(ux)$$

(1) This result is immediate from Theorem 24 of Hardy and Riesz (2).

and

$$k \int_u^\infty p^{-1} e^{-px} \overline{C}_\lambda^k(p)\, dp = o\left[\int_u^\infty p^{-1} e^{-px}\, dp\right] = o\{(ux)^{-1} e^{-ux}\}.$$

If we choose $ux = 1$, we see that

$$C_\lambda^k(u) - F_k(x) = o(1).$$

Hence $\sum_{n=0}^{\infty} a_n$ is summable (R, λ_n, k) to the sum s. We now deduce the result with the aid of (3.4) and Theorem 2.2.

References

[1] A. Amir (1952) - *On a converse of Abel's theorem*, « Proc. American Math. Soc. », *3*, 244–256.

[2] G. H. Hardy and M. Riesz (1915) - *The general theory of Dirichlet's series*, Cambridge Tract in Mathematics and Mathematical Physics, N. 18, Cambridge.