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p-Injectors and finite supersoluble groups

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Algebra. — *p-Injectors and finite supersoluble groups* (*). Nota (**)
di ANNA LUISA GILOTTI e LUIGI SERENA, presentata dal Socio G. ZAPPA.

RIASSUNTO. — In questa Nota gli Autori, proseguendo lo studio iniziato in [2] sui p-iniettori nei gruppi finiti, introducono la definizione di p-I-catena e provano che l'esistenza in un gruppo finito G di p-I-catene per ogni primo p che divide $|G|$ equivale alla supersolubilità di G.

Given a finite group G, a p-subgroup V of G is said to be a p-injector if the family

$$\mathbf{F}(V) = \{S , S \subseteq V^x , x \in G\}$$

is closed with respect to normal product; that is if $A , B \in \mathbf{F}(V)$ and $A , B \trianglelefteq AB$, then $AB \in \mathbf{F}(V)$.

Trivial examples of p-injectors are given by the Sylow p-subgroups and the normal p-subgroups. In [2] the Authors study general properties of p-injectors and give a characterization of them with respect to the class of p-soluble groups. In this Note, a necessary and sufficient condition is given for the supersolubility of a finite group in terms of p-injectors. More precisely after introducing the definition of p-I-chain, it is proved that the existence of such p-I-chains in G for any prime p which divides the order of G is equivalent to the supersolubility of G.

In the proof of the principal theorem, it is fundamental that there exists a p-I-chain relative to the smallest prime p which divides the order of G. In fact, if there is such a chain, then G will not be simple, since there is a p-normal complement.

All the groups considered will be finite.

1.

Let G be a group and let $\mathbf{F}_p$ be a family of p-subgroups of G which satisfies the conditions:

i) If $H \trianglelefteq S , S \in \mathbf{F}_p$ then $H \in \mathbf{F}_p$;

ii) If $A , B \in \mathbf{F}_p$ and $A , B \trianglelefteq AB$ then $AB \in \mathbf{F}_p$;

iii) If $S \in \mathbf{F}_p$ then $S^x \in \mathbf{F}_p$ for any $x \in G$.

$\mathbf{F}_p$ will be called a p-Fitting set.

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If N is a subnormal subgroup of G we write $\mathbf{F}_{p|_N}$ for the set of those subgroups of N which lie in $\mathbf{F}_p \cdot \mathbf{F}_{p|_N}$ is a p-Fitting set of N.

Let **A** be a set of p-subgroups of G and let $V \in \mathbf{A}$. V will be called **A**-maximal if no element of **A** contains V properly. V will be said to be an **A**-injector if V is **A**-maximal and for any $N \trianglelefteq\trianglelefteq G$, $V \cap N$ is $\mathbf{A}_{|_N}$-maximal.

A p-subgroup V of G is said to be a p-injector if there exists a p-Fitting set $\mathbf{F}_p$ such that V is a $\mathbf{F}_p$-injector.

The equivalence with the definition given in the introduction comes from the following proposition whose proof is in [2].

PROPOSITION 1.1. *Let* V *be a p-subgroup of* G, *then* V *is a p-injector if and only if the family* $\mathbf{F}(V) = \{S, S \subseteq V^x, x \in G\}$ *is closed with respect to normal product.*

Now we recall other propositions whose proofs are in [2].

PROPOSITION 1.2. *Let* V *be a p-injector and let* P *be a p-Sylow subgroup of* G *such that* $V \subseteq P$, *then* V *is weakly closed in* P *with respect to* G;

PROPOSITION 1.3. *Let* V *be a p-injector of* G, *then*:

i) *If* H *is a subgroup of* G *such that* $V \subseteq H$, *then* V *is a p-injector of* H.

ii) *If* $N \trianglelefteq G$ *then* $V \cap N$ *is a p-injector of* G.

PROPOSITION 1.4. *Let* G *be a p-soluble group and let* V *be a p-subgroup of* G. *Then* V *is a p-injector of* G *is and only if* $V = P \cap N$, *where* N *is a normal subgroup of* G *and* P *is a p-Sylow subgroup of* G.

In the following, we will use the next proposition the proof of which follows that of Anderson ([1] Proposition 2.2).

PROPOSITION 1.5. *Let* **F** *be a p-Fitting set of* G *and let* A *be a normal subgroup of* G, *then the family* $\overline{\mathbf{F}} = \{SA/A, S \in \mathbf{F}\}$ *is a p-Fitting set of* G/A.

In the proof we need the following:

LEMMA. *A set* **F** *of p-subgroups of* G *is a p-Fitting set if and only if* **F** *is invariant with respect to inner automorphisms and any subgroup* H *of* G *contains* $\mathbf{F}_{|H}$-*injectors.*

Proof. $\Rightarrow$ trivial.

$\Leftarrow$ It is sufficient to prove that for any $A \in \mathbf{F}$ and $N \trianglelefteq A$ then $N \in \mathbf{F}$ and if $A, B \in \mathbf{F}$ with $A, B \trianglelefteq AB$ then $AB \in \mathbf{F}$.

Let $A \in \mathbf{F}$ and $N \trianglelefteq A$. By hypothesis, A contains $\mathbf{F}_{|A}$-injectors.

Since $A \in \mathbf{F}$, A is a $\mathbf{F}_{|A}$-injector. It follows that $N = A \cap N$ is $\mathbf{F}_{|N}$-maximal. Therefore $N \in \mathbf{F}$. Now let $A, B \in \mathbf{F}$ with $A, B \trianglelefteq AB = M$. M contains $\mathbf{F}_{|M}$-injectors. Let V be one of them. Since $A \trianglelefteq M$, $B \trianglelefteq M$ it follows that $V \cap A$ and $V \cap B$ are $\mathbf{F}_{|A}$ and $\mathbf{F}_{|B}$-maximal respectively.

We have $V \cap A \subseteq A \in \mathbf{F}$ and $V \cap B \subseteq B \in \mathbf{F}$. Therefore $V \cap A = A$ and $V \cap B = B$ and then $V \supseteq (V \cap A)(V \cap B) = AB$. Since $V \subseteq AB$ it follows that $V = AB \in \mathbf{F}$.

Proof of Proposition 1.5. We indicate by " — " the natural homomorphism: $G \to G/A$ in such way that $\bar{G}$ means G/A and $\bar{H}$ means the homomorphic image of a subgroup H of G under " — ".

To prove that $\bar{\mathbf{F}}$ is a p-Fitting set of $\bar{G}$ it is sufficient to show that $\bar{\mathbf{F}}$ is closed with respect to inner automorphisms, and for any subgroup $\bar{H}$ of $\bar{G}$, $\bar{H}$ contains $\bar{\mathbf{F}}_{|\bar{H}}$-injectors. It is obvious that $\bar{\mathbf{F}}$ is closed with respect to the inner automorphisms. We prove the proposition by induction on $|G|$.

Let $\bar{H}$ be a proper subgroup of $\bar{G}$. We consider the family

$$\mathbf{F}_{|H} = \{S , S \subseteq H , S \in \mathbf{F}\} .$$

The $\mathbf{F}_{|H}$ is a p-Fitting set of H. Since $A \trianglelefteq H$ and $|H| \lneq |G|$, by induction $\overline{\mathbf{F}_{|H}} = \{\bar{S} , S \in \mathbf{F}_{|H}\}$ is a p-Fitting set of $\bar{H}$.

It follows that in $\bar{H}$ there is a $\bar{\mathbf{F}}_{|H}$-maximal element and then a $\mathbf{F}_{|H}$-injector. But $\overline{\mathbf{F}_{|H}} = \bar{\mathbf{F}}_{|\bar{H}} = \{\bar{S} \subseteq \bar{H} , S \in \mathbf{F}\}$ therefore $\bar{V}$ is $\bar{\mathbf{F}}_{|\bar{H}}$-injector. It remains to prove that $\bar{G}$ contains $\bar{\mathbf{F}}$-injectors.

Let V be an $\mathbf{F}$-injector of G. We prove that $\bar{V}$ is an $\bar{\mathbf{F}}$-injector of $\bar{G}$. Since V is $\mathbf{F}$-maximal in G, and since the maximal elements of $\mathbf{F}$ are conjugate, we have that $\bar{V}$ is $\bar{\mathbf{F}}$-maximal in $\bar{G}$. In fact, if we suppose $\bar{V} \subsetneqq \bar{W}$ with $\bar{W} \in \bar{\mathbf{F}}$, then there is a maximal element of $\mathbf{F}$ which contains W, say V^x with $x \in G$.

It follows $\bar{V} \subsetneqq \bar{W} \subset \bar{V}^x = (\bar{V})^{\bar{x}}$ and this is a contraddiction.

It remains to prove that, for any $\bar{N} \trianglelefteq\trianglelefteq \bar{G}$, $\bar{V} \cap \bar{N}$ is $\bar{\mathbf{F}}_{|\bar{N}}$-maximal.

We have $\bar{V} \cap \bar{N} = \overline{V \cap N}$. $V \cap N$ is an injector of N, and therefore $V \cap N$ is $\mathbf{F}_{|N}$-maximal. But any maximal element of $\mathbf{F}_{|N}$ is conjugate with $V \cap N$. It follows that $\bar{V} \cap \bar{N} = \overline{V \cap N}$ is $\bar{\mathbf{F}}_{|\bar{N}}$-maximal.

REMARK. We observe that if V is a p-injector of G, then $\bar{V}$ is a p-injector of $\bar{G}$.

2.

DEFINITION 2.1. *Let* G *be a finite group and let* P *be a* p*-Sylow subgroup of* G. *We say that* G *has a* p-I*-chain if there is a chain of the type*:

$$\langle 1 \rangle \lhd P_0 \lhd P_1 \lhd \cdots \lhd P_r = P \qquad \textit{with} \quad |P_i/P_{i-1}| = p$$

and where P_i *is a* p*-injector of* G.

THEOREM 2.2. *Let* G *be a* p*-soluble group, then* G *is* p*-supersoluble if and only if* G *has a* p-I*-chain.*

Proof. Let G be p-supersoluble, then G contains a chain of normal subgroups

$$\langle 1 \rangle = N_0 \lhd N_1 \lhd N_2 \lhd \cdots \lhd N_l = G ,$$

where N_i/N_{i-1} ($i = 1, \cdots, l$) is either a p'-group or a p-group of order p. Let P be a p-Sylow subgroup. If N_i/N_{i-1} is cyclic of order p, then $N_i \cap P/N_{i-1} \cap P$ is cyclic of order p, since $N_i \cap P$ and $N_{i-1} \cap P$ are p-Sylow subgroups of N_i and N_{i-1} respectively. If N_i/N_{i-1} is a p'-group, the $N_i \cap P = N_{i-1} \cap P$.

Now we consider the chain of subgroups of P:

$$\langle 1 \rangle = N_0 \cap P \trianglelefteq N_1 \cap P \trianglelefteq \cdots \trianglelefteq N_l \cap P = P .$$

The factors of such a chain are either cyclic of order p or the identity group. If we take away the repetitions, then we have a chain

$$\langle 1 \rangle = P_0 \triangleleft P_1 \triangleleft \cdots \triangleleft P_r = P$$

where $| P_i/P_{i-1} | = p$.

P_i ($i = 1, \cdots, r$) is a p-injector by Proposition 1.4.

Viceversa, let

$$\langle 1 \rangle = P_0 \triangleleft P_1 \triangleleft \cdots \triangleleft P_r = P$$

a p-I-chain of G.

We procede by induction on $|G|$.

Suppose $O_{p'}(G) \neq \langle 1 \rangle$; we indicate by $\bar{G} = G/O_{p'}(G)$ and $\bar{P}_i = P_i O_{p'}(G)/O_{p'}(G)$. Then the chain of $\bar{P}$

$$\langle 1 \rangle = \bar{P}_0 \triangleleft \bar{P}_1 \triangleleft \cdots \triangleleft \bar{P}_r = \bar{P}$$

is a p-I-chain of $\bar{G}$, since $\bar{P}_i$ is a p-injector of $\bar{G}$ and

$$\bar{P}_i/\bar{P}_{i-1} = \frac{P_i O_{p'}(G)/O_{p'}(G)}{P_{i-1} O_{p'}(G)/O_{p'}(G)} \cong P_i/P_{i-1}$$

is cyclic of order p.

By induction, $\bar{G}$ is p-supersoluble, therefore G is p-supersoluble.

If $O_{p'}(G) = \langle 1 \rangle$, then $O_p(G) \neq \langle 1 \rangle$. We consider P_1 ($| P_1 | = p$).

Since $\mathscr{C}_G(O_p(G)) \subseteq O_p(G)$, $P_1 \cap O_p(G) = P_1$, so P_1 is subnormal and pronormal in G, therefore it is normal in G.

Now, let $\bar{G} = G/P_1$ and $\bar{P}_i = P_i/P_1$ ($i = 2, \cdots, r$).

In $\bar{G}$ there is a p-I-chain. By induction, $\bar{G}$ is p-supersoluble, therefore G is p-supersoluble.

COROLLARY 2.3. *Let* G *be a finite soluble group. Then* G *is supersoluble if and only if* G *contains* p-I-*chains for any prime* p *which divides* $|G|$.

3.

We observe that the Theorem 2.2 can't be extended to the class of finite groups for any prime p. In fact, if we consider A_5 (the alternating group of degree 5), then there are a 3-I-chain and a 5-I-chain, but A_5 is neither 3-supersoluble nor 5-supersoluble.

But, for the smallest prime which divides $|G|$, we have the following:

THEOREM 3.1. *Let* G *be a finite group and let* p *be the smallest among the prime divisors of* $|G|$. *Then* G *has a* p*-normal complement if and only if* G *contains a* p-I-*chain.*

Proof. Suppose that G contains a p-normal complement K. Then G is p-supersoluble, so, by Theorem 2.2, G contains a p-I-chain.

Now let P be a p-Sylow subgroup of G relative to the smallest prime which divides $|G|$ and let

$$\langle 1 \rangle = P_0 \lhd P_1 \lhd \cdots \lhd P_r = P$$

a p-I-chain of G. We proceed by induction on $|G|$. Since $|P_1| = p$ then $P_1 \subseteq Z(P)$. P_1 is weakly closed in G by Proposition 1.2.

By Grun's theorem, we have

$$P \cap G' = P \cap H'. \qquad \text{Where} \quad H = \mathcal{N}_G(P_1).$$

Since $H = \mathcal{N}_G(P_1) \supseteq \mathcal{N}_G(P) \supseteq P$ it follows, by Proposition 1.3, that $\langle 1 \rangle = P_0 \lhd P_1 \lhd \cdots \lhd P_r = P$ is a p-I-chain of H, because P_i $(i = 1, \cdots, r)$ is a p-injector of H.

If $H = G$, then P_1 is normal in G. Let $\bar{G} = G/P_1$ and $\bar{P}_i = P_i/P_1$ $(i = 1, \cdots, r)$. The chain

$$\langle 1 \rangle = \bar{P}_1 \lhd \bar{P}_2 \lhd \cdots \lhd \bar{P}_r = \bar{P}$$

is a p-I-chain of $\bar{G}$. By induction, $\bar{G}$ contains a p-normal complement $\bar{K}$, that is $\bar{G} = \bar{K}\bar{P}$ and $\bar{K} \cap \bar{P} = \langle 1 \rangle$.

If $\bar{K} = K/P_1$, it follows $KP = G$ and $K \cap P = P_1$.

Then $|K| = pm$ with $(p, m) = 1$. Since p is the smallest prime dividing $|K|$, K contains a p-normal complement which is a p-normal complement of G. Let now $H \subsetneq G$. By induction H contains a p-normal complement Q; that is $H = QP$, $Q \cap P = \langle 1 \rangle$.

Since $Q \trianglelefteq H$, there is a normal subgroup in H of index a power of p; then $H' \cap P \neq P$. It follows $G' \cap P = H' \cap P \neq P$, therefore in G there is a normal subgroup such that the factor group is an abelian group of order a power of p. There is therefore a normal subgroup M of G whose index is p. $M \cap P$ is a p-Sylow subgroup of M and it contains the chain:

$$\langle 1 \rangle = M \cap P_0 \trianglelefteq M \cap P_1 \cdots \trianglelefteq M \cap P_r = M \cap P.$$

For any $i = 0, 1, \cdots, r-1$, we have either $|M \cap P_{i+1} : M \cap P_i| = p$ or $M \cap P_{i+1} = M \cap P_i$. Taking away the repetitions, we have a chain:

$$\langle 1 \rangle = K_0 \lhd K_1 \cdots \lhd K_s = M \cap P \quad \text{where} \quad |K_i/K_{i-1}| = p \quad (i = 1, \cdots, s)$$

and K_i is a p-injector of M, by Proposition 1.3.

It follows, by induction, that M contains a p-normal complement L which is a p-normal complement of G.

COROLLARY 3.2. *A finite group* G *is supersoluble if and only if it contains* p-I-*chains for any prime* p *which divides* $|G|$.

Proof. Let G be supersoluble; then, by Theorem 2.2, it contains such chains. On the other hand, let p be the smallest prime dividing $|G|$. By Theorem, 3.1, G contains a p-normal complement K. Since K contains q-I-chains for any prime q which divides $|K|$, proceeding by induction K is supersoluble; therefore G is soluble. By Corollary 2.3, G is supersoluble.

REFERENCES

[1] W. ANDERSON (1975) – *Injectors in finite solvable groups*, « Journal of Algebra », *36*, 333–338.
[2] A. L. GILOTTI e L. SERENA – *p–Iniettori nei gruppi finiti p–risolubili*, « Rend. Acc. Naz. dei Lincei » (to appear).
[3] POWELL–HIGMAN (1971) – *Finite simple groups*. Academic Press.