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Minimal prescription for matter terms in the gravitational theory

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Teorie relativistiche. — *Minimal prescription for matter terms in the gravitational theory* (*). Nota di GIANCARLO SPINELLI, presentata (**) dal Socio C. CATTANEO.

RIASSUNTO. — È noto che una teoria gravitazionale può essere costruita con un approccio di teoria dei campi nello spazio tempo pseudoeuclideo ove il potenziale gravitazionale viene rappresentato con un tensore doppio simmetrico $\psi_{\alpha\beta}$. Poichè tale approccio ha carattere iterativo, si presenta il problema della convergenza. Deser ha dimostrato che i termini di puro campo convergono ai corrispondenti della relatività generale. Nel presente lavoro si mostra come i termini della densità lagrangiana della teoria esatta, relativi alla materia ed alla sua interazione con il campo, si possano ottenere con un metodo di minima prescrizione a partire dalla densità lagrangiana di ordine zero. In tal maniera si ottengono i corrispondenti termini della relatività generale.

1. INTRODUCTION

It is well known [1, 2] that gravitation can also be treated as a usual field theory starting from the flat space-time. A symmetric tensor $\psi_{\alpha\beta}$ represents the gravitational potential in the pseudo-Euclidean " unrenormalized " [1] space-time (i.e. the space measured by ideal clocks and rods unaffected by gravity). Real rods and clocks are affected by $\psi_{\alpha\beta}$ and one can alternatively describe the motion measured by such real rods and clocks thinking of them as unaltered but so obtaning a curved space-time.

The fact is that such theories can be constructed only in an iterative form. One of the major problems is to show the convergence of the method and obtain the exact theory to which it converges. It was shown by Ogievetsky and Polubarinov [3] and independently by Wyss [4] that such a procedure converges to general relativity, imposing gauge invariance to all orders and only for the pure field terms. In a fundamental paper [5] Deser has shown the convergence to general relativity in a more general case and also in the presence of matter. As to the pure field terms Deser implements a linear action integral written in the Palatini form. He observes that taking as initial variables the contravariant components of the fundamental metric tensor, the exact action integral is reached at the third step of the iteration. As to the matter part (pure matter term and interaction terms between matter and gravitation) he uses the argument of the minimal prescription.

The minimal prescription for matter terms is here treated explicitly and the convergence to the relevant term of general relativity shown. The thing is of interest also for a future application to the pure field terms in order to

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have a general procedure to obtain the exact theory from the iteration even in those cases where the lucky circumstance of the end of the iteration procedure at the third step only, does not take place.

2. THE ITERATION PROCEDURE

The usual flat space-time iterative procedure is here briefly reported for reader's convenience and for stating the problem.

We consider an "unrenormalized" [1] pseudo-Euclidean space-time (i.e. the space measured by ideal clocks and rods unaffected by gravity). Gravity is represented by a symmetric tensor potential $\psi_{\alpha\beta}$. The theory deals with test particles whose coordinates are z^{α} in a generic reference frame of such flat space-time. The Dicke framework is accepted and then the theory has to be Lagrangean [6].

By varying the gravitational tensor potential $\psi_{\alpha\beta}$ in the action integral I, and equating to zero,

$$\underset{(\psi)}{\delta} I = 0 , \tag{1}$$

the field equations are obtained.

By varying the dynamical variables and equating to zero,

$$\underset{(z)}{\delta} I = 0 , \tag{2}$$

the equations of motion are obtained.

The action integral is:

$$I = \int d^4 x \, (-a)^{1/2} \, L , \tag{3}$$

where x^{α} are the coordinates of the generic point of the pseudo-Euclidean space-time, a is the determinant of the fundamental metric tensor $a_{\alpha\beta}$ of the pseudo-Euclidean space-time, and L is the Lagrangean density. The latter can be seen as the sum of a part relevant to pure field terms L_F and a part relevant to matter plus interaction L_M:

$$L = L_F + L_M . \tag{4}$$

Taking into account that in this kind of theories the maximum order of the derivatives of $\psi_{\alpha\beta}$ is the second one, eq. (1) is equivalent to

$$\frac{\partial L}{\partial \psi_{\alpha\beta}} - \left(\frac{\partial L}{\partial \psi_{\alpha\beta;\gamma}} \right)_{;\gamma} + \left(\frac{\partial L}{\partial \psi_{\alpha\beta;\gamma\lambda}} \right)_{;\gamma\lambda} = 0 , \tag{5}$$

where semicolons stand for covariant differentiation.

Here we consider only the case of matter made out of incoherent point-like particles. Taking into account that in the pseudo-Euclidean space-time the components of the four velocity of a generic particle are subjected to the constraint

$$\dot{z}^{\alpha}\,\dot{z}_{\alpha} = 1\,, \tag{6}$$

(where $\dot{z}^{\alpha} = \mathrm{d}z^{\alpha}/\mathrm{d}s$, $\mathrm{d}s^2 = a_{\alpha\beta}\,\mathrm{d}z^{\alpha}\,\mathrm{d}z^{\beta}$, and we put the light speed $c = 1$), eq. (2) is equivalent to [7, 8]

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial \mathrm{L}}{\partial \dot{z}^{\alpha}} - \frac{\partial \mathrm{L}}{\partial z^{\alpha}} = \frac{\mathrm{D}}{\mathrm{d}s}\left[\left(\frac{\partial \mathrm{L}}{\partial \dot{z}^{\beta}}\,\dot{z}^{\beta} - \mathrm{L}\right)\dot{z}_{\alpha}\right], \tag{7}$$

where D/ds denotes covariant differentiation.

At first order (in the coupling constant f) the well known Lagrangean densities are

$$\mathrm{L}_{\mathrm{F}}^{(1)} = \tfrac{1}{2}\,\psi_{\alpha\beta;\gamma}\,\psi^{\alpha\beta;\gamma} - \psi_{\alpha\beta;\gamma}\,\psi^{\alpha\gamma;\beta} + \psi_{\alpha\beta}{}^{;\beta}\,\psi^{;\alpha} - \tfrac{1}{2}\,\psi_{;\alpha}\,\psi^{;\alpha}\,, \tag{8}$$

and

$$\mathrm{L}_{\mathrm{M}}^{(1)} = \frac{\Sigma_q}{\sqrt{-a}}\int \mathrm{d}t\,\Lambda_{(q)}^{(1)}\,\delta^4\,(x - z_{(q)}) = \tag{9}$$

$$= \frac{\Sigma_q}{\sqrt{-a}}\int \mathrm{d}s\,\delta^4\,(x - z_{(q)})\,m_{(q)}\,(f\psi_{\alpha\beta} - a_{\alpha\beta})\,\dot{x}^{\alpha}\,\dot{x}^{\beta}\,.$$

where $m_{(q)}$ is the proper mass and $z_{(q)}$ the coordinates of the q-th particle of the considered incoherent matter and, in general, $\Lambda_{(q)}^{(n)}$ is defined by $\mathrm{L}_{\mathrm{M}}^{(n)} = (-a)^{-1/2}\,\Sigma_q \int \Lambda_{(q)}^{(n)}\,\delta^4\,(x - z_{(q)})\,\mathrm{d}t$, t being an auxiliary integration variable.

Putting eqs. (8) and (9) into eqs. (5) gives the first order field equations

$$\Box\psi^{\alpha\beta} - \psi_{\sigma}{}^{(\alpha;\beta)\sigma} + \psi^{;\alpha\beta} + a^{\alpha\beta}\,(\psi^{\sigma\lambda}{}_{;\sigma\lambda} - \Box\psi) = f\mathrm{T}^{(p)\alpha\beta}\,. \tag{10}$$

where

$$\mathrm{T}^{(p)\alpha\beta} = \frac{1}{\sqrt{-a}}\sum_{(q)} m_{(q)}\int \mathrm{d}s\,\delta^4\,(x - z_{(q)})\,\dot{x}^{\alpha}\,\dot{x}^{\beta}\,, \tag{11}$$

and where $\Box\psi_{\alpha\beta} = \psi_{\alpha\beta;\lambda}{}^{\lambda}$ and parentheses containing two indices denote symmetrization.

Putting eqs. (8) and (9) into eqs. (7) gives the first order equations of motion

$$[(1 + f\psi_{\mu\nu}\,\dot{z}_{(q)}^{\mu}\,\dot{z}_{(q)}^{\nu})\,\dot{z}_{(q)\alpha} - 2f\psi_{\alpha\lambda}\,\dot{z}_{(q)}^{\lambda}]_{;\gamma}\,\dot{z}_{(q)}^{\gamma} = -f\psi_{\gamma\lambda;\alpha}\,\dot{z}_{(q)}^{\gamma}\,\dot{z}_{(q)}^{\lambda}\,, \tag{12}$$

for the q-th particle.

Eqs. (10) and (12) are not consistent. Indeed the left hand side (LHS) of eqs. (10) is divergenceless, thus the same must happen for the RHS. But this is in contrast with eqs. (12). To overcome this drawback one takes as second order field equations eqs. (10) in which the RHS has been substituted by the total energy-momentum tensor $T^{\alpha\beta}$, i.e. a symmetric tensor whose divergence equated to zero gives eqs (12). It can be shown that a particular energy-momentum tensor can be obtained [9, 10]as

$$(13) \qquad \tilde{T}_{\alpha\beta} = \frac{2}{\sqrt{-a}} \frac{\delta(\sqrt{-a}\,L)}{\delta a^{\alpha\beta}} = \\ = \frac{2}{\sqrt{-a}} \left[\frac{\partial(\sqrt{-a}\,L)}{\partial a^{\alpha\beta}} - \left(\frac{\partial(\sqrt{-a}\,L)}{\partial a^{\alpha\beta}{}_{,\gamma}} \right)_{;\gamma} \right].$$

It is given explicitly in eqs. (13) of Ref. [10] (where the presence of an electromagnetic field is also considered). But it is not unique [11], since it can be implemented by the most general second order divergenceless tensor $t_{\alpha\beta}$ containing 5 arbitrary parameters (4 if one requires the theory to be Lagrangean). Eventually we have:

$$(14) \qquad T_{\alpha\beta} = \tilde{T}_{\alpha\beta} + t_{\alpha\beta} .$$

This is the tensor to be substituted for $T^{(p)}_{\alpha\beta}$ at the RHS of (10) in order to have the second order field equations.

Now one can get the second order action integral $I^{(2)}$ as the one which inserted in (1) gives the second order field equations. Then, by (7), one gets the second order equations of motion. Again these two sets of equations are not consistent as it happened for the first order. One repeats the procedure and the method becomes iterative. The same happens also if one assumes the Hilbert gauge to all orders [12].

3. Convergence (to general relativity)

As to the pure field terms it was shown by Deser [5] that the exact action integral is reached at the third step of the iteration. He obtains such result by writing the first order action integral in the Palatini form and taking as "initial variables" the contravariant components of the fundamental metric tensor. The same result has been obtained taking into account all the arbitrariness of the energy-momentum tensor, both assuming the Hilbert gauge [12] and without such limitation [11].

As to the matter part (pure matter and interaction terms between matter and gravitational potential) the argument of minimal prescription is used [5]. A deduction of such minimal prescription for the matter terms is here given.

Let L be the exact Lagrangean density to which the theory converges, and $L^{(n)}$ the Lagrangean density of the n-th step.

As we have seen in the preceding section $\delta L^{(n)}/\delta\psi_{\alpha\beta} = 0$ gives the n-th order field equations. The same equations can also be obtained starting from the $(n-1)$-th order field equations by substituting in the RHS of them $T^{(n)}_{\alpha\beta} = 2(-a)^{-1/2}\,\delta(L^{(n-1)}\sqrt{-a})/\delta a^{\alpha\beta}$ for $T^{(n-1)}_{\alpha\beta}$. We are now considering only the matter terms of such equations, that is terms in which the matter appears either alone or coupled with the gravitational field. Such terms appear only at the RHS of field equations written in the form of eqs. (10). Thus, if we call $L^{(n)}_{M}$ the part of the n-th order Lagrangean density relevant to matter, our procedure implies

$$f\,\frac{2}{\sqrt{-a}}\,\frac{\delta\,(L^{(n-1)}_{M}\sqrt{-a})}{\delta a_{\alpha\beta}} = -\,\frac{\delta L^{(n)}_{M}}{\delta\psi_{\alpha\beta}}\,. \tag{15}$$

The minus sign comes from the fact that $\delta L^{(n)}_{M}/\delta\psi_{\alpha\beta}$ gives the matter terms of the field equations written at the LHS of them, i.e. it gives $-fT^{(n)\alpha\beta}$.

Now it can be noticed that $L^{(n)}_{M}$, because of its structure, does depend neither on the derivatives of $a_{\alpha\beta}$ nor on the derivatives of $\psi_{\alpha\beta}$. (In passing we can notice that none of these lucky circumstances does happen for the other part of the Lagrangean density, that is for pure field terms). This fact is taken into account by using the symbols $\partial/\partial a_{\alpha\beta}$ and $\partial/\partial\psi_{\alpha\beta}$ for the functional derivatives $\delta/\delta a_{\alpha\beta}$ and $\delta/\delta\psi_{\alpha\beta}$. Hence, eqs. (15) can be written as

$$\frac{\partial\,(L^{(n-1)}_{M}\sqrt{-a})}{\partial a_{\alpha\beta}} = \frac{\partial\,(L^{(n)}_{M}\sqrt{-a})}{\partial h_{\alpha\beta}}\,, \tag{16}$$

where $h_{\alpha\beta} = -2f\psi_{\alpha\beta}$. Because of the structure of L_M [see eqs. (9)], and because it cannot simultaneously be $z^{\alpha}_{(q)} = z^{\alpha}_{(s)}$ for every α if $q \neq s$, eqs. (16) imply

$$\frac{\partial\Lambda^{(n-1)}_{(q)}}{\partial a_{\alpha\beta}} = \frac{\partial\Lambda^{(n)}_{(q)}}{\partial h_{\alpha\beta}}\,, \tag{17}$$

where $\Lambda^{(n)}_{(q)}$ is calculated with $x^{\alpha}_{(q)} = z^{\alpha}$. $\Lambda^{(n)}_{(q)}$ is a function of $a_{\alpha\beta}(\boldsymbol{z})$ and of $h_{\alpha\beta}(\boldsymbol{z})$. Equations (17) are ten for each n and q ($\alpha, \beta = 0, 1, 2, 3$).

To the set of conditions (17) one must add that $\Lambda^{(0)}_{(q)}$ is the exact expression of $\Lambda_{(q)}$ when no gravitational field is present, i.e.

$$\Lambda^{(0)}_{(q)}(\boldsymbol{a}, \boldsymbol{h}) = \Lambda_{(q)}(\boldsymbol{a}, 0) = \varphi_{(q)}(\boldsymbol{a})\,. \tag{18}$$

The symbols $\boldsymbol{a}$ and $\boldsymbol{h}$ denote that $\Lambda^{(0)}_{(q)}$ depends on all the components $a_{\alpha\beta}$ and $h_{\alpha\beta}$; the function $\varphi_{(q)}(\boldsymbol{a})$ is introduced for shortening the notation.

Eliminating the gravitational field gives, to every order of approximation, the zeroth order approximation; we have therefore to add the condition

$$\Lambda^{(n)}_{(q)}(\boldsymbol{a}, 0) = \Lambda_{(q)}(\boldsymbol{a}, 0) = \varphi_{(q)}(\boldsymbol{a}). \tag{19}$$

For sake of simplicity, let us first consider one particle only and the monodimensional case. Here the eqs. (17), (18) and (19) reduce to

$$\frac{\partial f^{(n)}}{\partial h} = \frac{\partial f^{(n-1)}}{\partial a}, \tag{20}$$

$$f^{(0)}(a, h) = f(a, 0) = \varphi(a), \tag{21}$$

$$f^{(n)}(a, 0) = f(a, 0) = \varphi(a), \tag{22}$$

where we have substituted $\Lambda^{(n)}_{(q)}$ by a function $f^{(n)}(a, h)$ with $a = a(z)$ and $h = h(z)$.

Equations (20) imply

$$f^{(n)}(a, h) = f^{(n)}(a, 0) + \int_0^h \frac{\partial f^{(n-1)}(a, \xi)}{\partial a}\, d\xi. \tag{23}$$

By eqs. (21), (22), and (23) we get

$$f^{(n)}(a, h) = \sum_0^n{}_j \frac{h^j}{j!}\, \varphi^{(j)}(a), \tag{24}$$

where $\varphi^{(j)}$ is the j-th order derivative of φ with respect to a. If the sequence $\{f^{(n)}\}$ converges uniformly, in a domain D, to a function $f(a, h)$, it is

$$f(a, h) = \varphi(a + h). \tag{25}$$

Let us now generalize to our case where $\boldsymbol{a}$ and $\boldsymbol{h}$ have both ten components, and eqs. (17) are ten for each n and q. By eqs. (17), (18), and (19) one can obtain

$$\Lambda^{(n)}_{(q)}(\boldsymbol{a}, \boldsymbol{h}) = \varphi_{(q)}(\boldsymbol{a}) + h_{\alpha\beta}\frac{\partial \varphi_{(q)}(\boldsymbol{a})}{\partial a_{\alpha\beta}} + \cdots + \left(h_{\alpha\beta}\frac{\partial}{\partial a_{\alpha\beta}}\right)^{(n)} \varphi_{(q)}(\boldsymbol{a}). \tag{26}$$

If the sequence $\{\Lambda^{(n)}_{(q)}\}$ uniformily converges to a function $\Lambda_{(q)}(a, h)$, eq. (26) implies

$$\Lambda_{(q)}(\boldsymbol{a}, \boldsymbol{h}) = \varphi_{(q)}(\boldsymbol{a} + \boldsymbol{h}) = \Lambda^{(0)}_{(q)}(\boldsymbol{a} + \boldsymbol{h}). \tag{27}$$

This is what one usually calls minimal prescription.

In our case of point-like particles it is

$$-\Lambda_{(q)}^{(0)} = m_{(q)} \left(a_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \right)^{1/2}. \tag{28}$$

By eq. (26) one gets

$$-\Lambda_{(q)}^{(n)} = m_{(q)} \sum_{0}^{n} {}_j \binom{1/2}{j} (h_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta)^j \left(a_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \right)^{1/2}. \tag{29}$$

The sequence $\{\Lambda_{(q)}^{(n)}\}$ uniformly (with respect $h_{\alpha\beta}$) converges for $x^\alpha = z_{(q)}^\alpha$ if $| h_{\alpha\beta}\, \dot{z}_{(q)}^\alpha\, \dot{z}_{(q)}^\beta| \leq 1$. Under this condition we have

$$-\Lambda_{(q)} = m_{(q)} \left[(a_{\alpha\beta} + h_{\alpha\beta}) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \right]^{1/2}. \tag{30}$$

Now we make the hypothesis that the exact $\Lambda_{(q)}$ of the theory is an analytic function of the $h_{\alpha\beta}$.

Under this hypothesis we can make an analytic continuation of $\Lambda_{(q)}$ on the real axis for $h_{\alpha\beta}\, \dot{z}_{(q)}^\alpha\, \dot{z}_{(q)}^\beta \geq -1$. By the identity principle for analytic functions, the two functions (the original and the continued one) will remain the same (given by eq. (30)) in the whole domain $h_{\alpha\beta}\, \dot{z}_{(q)}^\alpha\, \dot{z}_{(q)}^\beta \geq -1$.

If the theory is reinterpreted [1, 5, 11] in a Riemannian space whose fundamental metric tensor in given by

$$g_{\alpha\beta} = a_{\alpha\beta} + h_{\alpha\beta}\,, \tag{31}$$

the exact $\Lambda_{(q)}$ (and hence the exact action integral I_M) in such space time is obtained by $\Lambda_{(q)}^{(0)}$ (or by $I_M^{(0)}$) of the flat space-time by the substitution $a_{\alpha\beta} \to g_{\alpha\beta}$. In such a way the matter term of general relativity is obtained:

$$I_M = -\sum_q \int m_{(q)} ds_{(q)}^*\,, \tag{32}$$

where the star denotes that $ds_{(q)}^*$ has been calculated in the curved space-time. The condition $h_{\alpha\beta}\, \dot{z}_{(q)}^\alpha\, \dot{z}_{(q)}^\beta \geq -1$ corresponds to $ds_{(q)}^{*2} \geq 0$, i.e. to having speeds lower than the light speed.

As to the pure field terms, the convergence to general relativity was proved by Deser [5]. Thus the equivalence of the two approaches to general relativity (the curved space-time approach and the field theory approach in the flat space-time) seems to be proved. The convenience of the use of one or of the other approach will only be technical and will depend on the particular problem to be treated.

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