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On concircular Bianchi and Veblen identities

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Geometria differenziale. — *On concircular Bianchi and Veblen identities.* Nota(*) di CHANDRA MANI PRASAD e RAVINDRA KUMAR SRIVASTAVA, presentata dal Socio B. SEGRE.

RIASSUNTO. — Si ottengono delle identità fra le componenti del tensore di curvatura concircolare (introdotto in [1] da K. Yano).

1. INTRODUCTION

Let V_n be a Riemannian space of class of any required order. The dimension n is greater than two, unless specifically mentioned. Let R^h_{ijk}, R_{ij} and R denote the curvature tensor, Ricci tensor and scalar curvature respectively. The concircular curvature tensor Z^h_{ijk} is given by

$$Z^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)} (\delta^h_k g_{ij} - \delta^h_j g_{ik}) . \tag{1.1}$$

This tensor Z^h_{ijk} may be contracted in two ways giving

$$Z^h_{hjk} = 0 , \tag{1.2}$$

and

$$Z^h_{ijh} = R_{ij} - \frac{R}{n} g_{ij} . \tag{1.3}$$

Let us call $Z_{ij} \overset{\text{def}}{=} Z^h_{ijh}$ the concircular Ricci tensor.

2. CONCIRCULAR BIANCHI IDENTITY

The Bianchi identities in a V_n are defined by

$$A^h_{ijkl} \overset{\text{def}}{=} R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k} = 0 \tag{2.1}$$

where, comma (,) denotes the covariant differentiation with respect to x's.

Differentiating (1.1) covariantly with respect to x^l we have

$$Z^h_{ijk,l} = R^h_{ijk,l} - \frac{R_{,l}}{n(n-1)} (\delta^h_k g_{ij} - \delta^h_j g_{ik}) . \tag{2.2}$$

On the analogy of the Bianchi identities for the curvature tensor of V_n, we define the corresponding expression for the concircular curvature tensor of V_n as follows

$$K^h_{ijkl} \overset{\text{def}}{=} Z^h_{ijk,l} + Z^h_{ikl,j} + Z^h_{ilj,k} . \tag{2.3}$$

(*) Pervenuta all'Accademia il 6 luglio 1977.

With the help of (2.1) and (2.2), equation (2.3) becomes

$$K^h_{ijkl} = A^h_{ijkl} - \frac{1}{n(n-1)} \{R_{,l} (\delta^h_k g_{ij} - \delta^h_j g_{ik}) + R_{,j} (\delta^h_l g_{ik} - \delta^h_k g_{il}) + R_{,k} (\delta^h_j g_{il} - \delta^h_l g_{ij})\} . \tag{2.4}$$

Contracting (2.4) for h and l and then multiplying the resulting expression by g^{ij} and on some simplifications, we have

$$R_{,k} = \frac{n}{(n-2)} g^{ij} (K^h_{ijkh} - A^h_{ijkh}) . \tag{2.5}$$

Substituting (2.5) in (2.4) and replacing δ^h_k by $g^{hm} g_{mk}$ etc. in the equation obtained, we have

$$\begin{aligned} K^h_{ijkl} &+ \frac{g^{pq} g^{hm}}{(n-1)(n-2)} \{(g_{mk} g_{ij} - g_{mj} g_{ik}) K^r_{pqlr} + (g_{ml} g_{ik} - g_{mk} g_{il}) K^r_{pqjr} + (g_{mj} g_{il} - g_{ml} g_{ij}) K^r_{pqkr}\} \\ &= A^h_{ijkl} + \frac{g^{pq} g^{hm}}{(n-1)(n-2)} \{(g_{mk} g_{ij} - g_{jm} g_{ik}) A^r_{pqlr} + (g_{ml} g_{ik} - g_{mk} g_{il}) A^r_{pqjr} + (g_{mj} g_{il} - g_{ml} g_{ij}) A^r_{pqkr}\} . \end{aligned} \tag{2.6}$$

Since the Bianchi identities in a V_n are satisfied, the right-hand side of the equation (2.6) is identically zero. We, therefore, have

$$K^h_{ijkl} + \frac{g^{pq} g^{hm}}{(n-1)(n-2)} \{(g_{mk} g_{ij} - g_{mj} g_{ik}) K^r_{pqlr} + (g_{ml} g_{ik} - g_{mk} g_{il}) K^r_{pqjr} + (g_{mj} g_{il} - g_{ml} g_{ij}) K^r_{pqkr}\} = 0 . \tag{2.7}$$

Consequently, this equation can be written as

$$\begin{aligned} T^h_{ijkl} = Z^h_{ijk,l} + Z^h_{ikl,j} + Z^h_{ilj,k} &+ \frac{g^{hm} g^{pq}}{(n-1)(n-2)} \{(g_{mk} g_{ij} - g_{mj} g_{ik}) \cdot \\ \cdot (Z^r_{pql,r} + Z^r_{plr,q} + Z^r_{prq,l}) &+ (g_{ml} g_{ik} - g_{mk} g_{il}) (Z^r_{pqj,r} + Z^r_{pjr,q} + Z^r_{prq,j}) + \\ &+ (g_{mj} g_{il} - g_{ml} g_{ij}) (Z^r_{pqk,r} + Z^r_{pkr,q} + Z^r_{prq,k})\} = 0 . \end{aligned} \tag{2.8}$$

We call (2.8), the concircular Bianchi identity and the tensor T^h_{ijkl} the concircular Bianchi Tensor.

LEMMA (2.1). *If the covariant derivative of the Ricci tensor is symmetric, the scalar curvature* R *is constant.*

Proof. From the given condition, we have,

$$R_{ij,k} = R_{kj,i} .$$

Multiplying this equation by g^{ij}, we have

$$g^{ij}\,R_{ij,k} = g^{ij}\,R_{kj,i} \qquad \text{or} \qquad R_{,k} = R^{i}_{k,i}\,.$$

But

$$R^{i}_{k,i} = \frac{1}{2}\,R_{,k} = \frac{1}{2}\,\frac{\partial R}{\partial x^{k}}\,, \qquad [2]$$

so

$$R_{,k} = 0\,.$$

THEOREM (2.1). *The concircular Bianchi identities and the Bianchi identities in a* V_n *are identical, if the covariant derivative of the Ricci tensor is symmetric.*

Proof. From the Lemma (2.1) and the equation (2.4), the proof follows.

3. CONCIRCULAR VEBLEN IDENTITY

The Veblen identities in a V_n are given by

$$\text{(3.1)} \qquad V^{h}_{ijkl} \overset{\text{def}}{=} R^{h}_{ijk,l} + R^{h}_{kil,j} + R^{h}_{lkj,i} + R^{h}_{jil,k} = 0\,.$$

Similarly, we define the concircular Veblen identities for the concircular curvature tensor of V_n as follows

$$\text{(3.2)} \qquad W^{h}_{ijkl} \overset{\text{def}}{=} Z^{h}_{ijk,l} + Z^{h}_{kil,j} + Z^{h}_{lkj,i} + Z^{h}_{jli,k}\,.$$

From the equation (2.2) substituting the values of $Z^{h}_{ijk,l}$ etc., we have

$$\text{(3.3)} \qquad W^{h}_{ijkl} = V^{h}_{ijkl} - \frac{1}{n\,(n-1)}\,\{R_{,l}\,(\delta^{h}_{k}\,g_{ij} - \delta^{h}_{j}\,g_{ik}) + R_{,j}\,(\delta^{h}_{l}\,g_{ik} - \delta^{h}_{i}\,g_{kl}) + $$
$$+ R_{,i}\,(\delta^{h}_{l}\,g_{lk} - \delta^{h}_{k}\,g_{lj}) + R_{,k}\,(\delta^{h}_{i}\,g_{jl} - \delta^{h}_{l}\,g_{ji})\}\,.$$

Contracting for h and l and on some simplifications, we have

$$\text{(3.4)} \qquad R_{,k} = \frac{n}{(n-2)}\,(g^{ij}\,W^{h}_{ijkh} - g^{ij}\,V^{h}_{ijkh})\,.$$

Substituting from (3.4) in (3.3) and replacing δ^{h}_{k} by $g^{hm}\,g_{mk}$ etc. and considering (1.2) whe have

$$\text{(3.5)} \qquad Z^{h}_{ijk,l} + Z^{h}_{kil,j} + Z^{h}_{lkj,i} + Z^{h}_{jli,k} + \frac{g^{hm}\,g^{pq}}{(n-1)(n-2)}\,\{(g_{mk}\,g_{ij} -$$
$$- g_{mj}\,g_{ik})\,(Z^{r}_{pql,r} + Z^{r}_{lpr,q} + Z^{r}_{qrp,l}) +$$
$$+ (g_{ml}\,g_{ki} - g_{mi}\,g_{kl})\,(Z^{r}_{pqj,r} + Z^{r}_{jpr,q} + Z^{r}_{qrp,j}) + (g_{mj}\,g_{lk} - g_{mk}\,g_{lj})\,\cdot$$
$$\cdot\,(Z^{r}_{pqi,r} + Z^{r}_{ipr,q} + Z^{r}_{qrp,i}) + (g_{mi}\,g_{jl} - g_{ml}\,g_{ji})\,(Z^{r}_{pqk,r} + Z^{r}_{kpr,q} + Z^{r}_{qrp,k})\}$$
$$= V^{h}_{ijkl} + \frac{g^{pq}\,g^{hm}}{(n-1)(n-2)}\,\{(g_{mk}\,g_{ij} - g_{mj}\,g_{ik})\,V^{r}_{pqlr} + (g_{ml}\,g_{ki} - g_{mi}\,g_{kl})\,\cdot$$
$$\cdot\,V^{r}_{pqjr} + (g_{mj}\,g_{lk} - g_{mk}\,g_{lj})\,V^{r}_{pqir} + (g_{mi}\,g_{jl} - g_{ml}\,g_{ji})\,V^{r}_{pqkr}\}\,.$$

Since the Veblen identities in a V_n are satisfied, the right hand side of the above equation is identically zero. Let us put the left hand side of (3.5) equal to P^h_{ijkl}, thus

$$(3.6) \qquad P^h_{ijkl} \overset{\text{def}}{=} Z^h_{ijk,l} + Z^h_{kil,j} + Z^h_{lkj,i} + Z^h_{jli,k} + \frac{g^{pq}\, g^{hm}}{(n-1)(n-2)}$$

$$\{(g_{mk}\, g_{ij} - g_{mj}\, g_{ik})\,(Z^r_{pql,r} + Z^r_{lpr,q} + Z^r_{qrp,l}) + (g_{ml}\, g_{ki} - g_{mi}\, g_{kl}) \cdot$$

$$\cdot (Z^r_{pqj,r} + Z^r_{jpr,q} + Z^r_{qrp,j}) + (g_{mj}\, g_{lk} - g_{mk}\, g_{lj})\,(Z^r_{pqi,r} + Z^r_{ipr,q} + Z^r_{qrp,i}) +$$

$$+ (g_{mi}\, g_{jl} - g_{ml}\, g_{ji})\,(Z^r_{pqk,r} + Z^r_{kpr,q} + Z^r_{qrp,k})\} = 0 .$$

DEFINITION (3.1). We call equation (3.6) the concircular Veblen identities and the tensor P^h_{ijkl} the concircular Veblen tensor.

From (3.6), we have the following

THEOREM (3.1). *In any Riemannian space* V_n $(n > 2)$, *the following concircular Veblen identities*

$$(3.7) \qquad P^h_{ijkl} = W^h_{ijkl} + Q^i_{ijkl}$$

hold, where Q^h_{ijkl} *is the sum of all the terms in the right hand side of* (3.6) *excluding the first four terms.*

We observe that the right hand side of (3.5) is also an identity; we call it the second form of the concircular Veblen identity and state the following

THEOREM (3.2). *In any Riemannian space* V_n $(n > 2)$, *the following concircular Veblen identities in terms of the tensor*

$$(3.8) \qquad P^h_{ijkl} = V^h_{ijkl} + S^h_{ijkl}$$

hold, where S^h_{ijkl} *is the right-hand side of* (3.5) *excluding the first term.*

THEOREM (3.3). *If the covariant derivative of the Ricci tensor is symmetric, the concircular Veblen identities and the Veblen identities are identical.*

Proof. From Lemma (2.1) and equation (3.2), the proof follows.

From the equations (3.6) and (2.8) we have

$$(3.9) \qquad P^h_{ijkl} + P^h_{iklj} + P^h_{iljk} = T^h_{ijkl} + T^h_{lkji} + T^h_{kilj} + T^h_{jlik} + B^h_{ijkl}$$

where

$$B^h_{ijkl} = 2\,(Z^h_{lij,k} + Z^h_{jli,k} + Z^h_{kjl,i} + Z^h_{lki,j} + \frac{g^{hm}\, g^{pq}}{(n-1)(n-2)}$$

$$\{(g_{mj}\, g_{il} - g_{mi}\, g_{jl})\,(Z^r_{pqk,r} + Z^r_{pkr,q} + Z^r_{prq,k}) + (g_{mi}\, g_{kl} - g_{mk}\, g_{il})$$

$$(Z^r_{pqj,r} + Z^r_{jpr,q} + Z^r_{qrp,j}) + (g_{mj}\, g_{kl} - g_{mk}\, g_{jl})\,(Z^r_{pqi,r} + Z^r_{pir,q} + Z^r_{prq,i})\} .$$

THEOREM (3.4). (3.9) *is the relation between the concircular Bianchi and Veblen identities defined in* (2.8) *and* (3.6) *respectively.*

THEOREM (3.5). *The necessary and sufficient condition that the concircular Bianchi and Veblen identities can be expressed explicitly in terms of one another is that the tensor* B^h_{ijkl} *is equal to zero.*

4. APPLICATION TO AN EINSTEIN SPACE

In this section, we shall study the properties of the concircular Bianchi and Veblen identities in an Einstein space V_n. It is well known that every V_2 is an Einstein space. An Einstein space V_3 is a spherical space of constant Riemannian curvature [4]. Therefore, we shall consider the case $n \geqq 4$.

Let us suppose that V_n is an Einstein space, therefore we have

$$R_{ij} = \frac{R}{n} g_{ij} . \tag{4.1}$$

It is obvious that the concircular Ricci tensor vanishes identically for an Einstein space according to equations (4.1) and (1.3).

For $n > 2$, equation (4.1) holds only if R is constant [3]. Therefore, R being constant, equations (2.4) and (3.3) give

$$K^h_{ijkl} = A^h_{ijkl} \tag{4.2}$$

and

$$W^h_{ijkl} = V^h_{ijkl} . \tag{4.3}$$

From (4.2) and (2.6) we have

$$\begin{aligned} \frac{g^{pq} g^{hm}}{(n-1)(n-2)} \{ & (g_{mk} g_{ij} - g_{mj} g_{ik}) (Z^r_{pql,r} + Z^r_{plr,q} + Z^r_{prq,l}) + \\ & + (g_{ml} g_{ik} - g_{mk} g_{il}) (Z^r_{pqj,r} + Z^r_{pjr,q} + Z^r_{prq,j}) + \\ & + (g_{mj} g_{il} - g_{ml} g_{ij}) (Z^r_{pqk,r} + Z^r_{pkr,q} + Z^r_{prq,k}) \} = 0 . \end{aligned} \tag{4.4}$$

Accordingly, from (4.3) and (3.5) we have

$$\begin{aligned} \frac{g^{hm} g^{pq}}{(n-1)(n-2)} \{ & (g_{mk} g_{ij} - g_{mj} g_{ik}) (Z^r_{pql,r} + Z^r_{lpr,q} + Z^r_{qrp,l}) + \\ & + (g_{ml} g_{ki} - g_{mi} g_{kl}) (Z^r_{pqj,r} + Z^r_{jpr,q} + Z^r_{qrp,j}) + \\ & + (g_{mj} g_{lk} - g_{mk} g_{lj}) (Z^r_{pqi,r} + Z^r_{ipr,q} + Z^r_{qrp,i}) + (g_{mi} g_{jl} - g_{ml} g_{ji}) \\ & (Z^r_{pqk,r} + Z^r_{kpr,q} + Z^r_{qrp,k}) \} = 0 . \end{aligned} \tag{4.5}$$

In a Riemannian space V_n the ordinary Bianchi and Veblen identities are satisfied, therefore from equations (4.2) and (4.3) we have

$$K^h_{ijkl} = 0 , \tag{4.6}$$

$$W^h_{ijkl} = 0 . \tag{4.7}$$

From (4.2) and (4.6) we establish the following

THEOREM (4.1). *The concircular Bianchi identities in an Einstein space and in a Riemannian space* V_n *are identical.*

THEOREM (4.2). *For an Einstein space* (4.4) *and* (4.6) *are identities and these are equivalent to the concircular Bianchi identities.*

Consequently, from (4.3) and (4.7) we have,

THEOREM (4.3). *The concircular Veblen identities in an Einstein space and in a Riemannian space are identical.*

THEOREM (4.4). *For an Einstein space* (4.5) *and* (4.7) *are identities and these are equivalent to the concircular Veblen identities.*

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