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Further Results on Asymptotic Behavior of Solutions of Functional Differential Equations

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Equazioni differenziali. — *Further Results on Asymptotic Behavior of Solutions of Functional Differential Equations.* Nota (*) di CHEH-CHIH YEH (**), presentata dal Socio G. SANSONE.

RIASSUNTO. — Si studia il comportamento asintotico delle soluzioni di un'equazione differenziale nonlineare

$$x^{(n)}(t) + \delta f(t, x[g_1(t)], \cdots, x[g_m(t)]) = h(t), \qquad \delta = \pm 1$$

sotto particolari ipotesi.

1. INTRODUCTION

The results of this note are inspired by a recent paper of Murakami-Nakagiri-Yeh [3]. In [3] we established the asymptotic behavior of solutions of the following differential equation with deviating arguments

$$x^{(n)}(t) + \delta\Phi(t) F(x[g_1(t)], \cdots, x[g_m(t)]) = h(t), \qquad \delta = \pm 1$$

by using nonstandard techniques.

The purpose of this paper is to establish some new criteria for the asymptotic behaviour of solutions of functional differential equations of the form

$$E(\delta) \qquad x^{(n)}(t) + \delta f(t, x[g_1(t)], \cdots, x[g_m(t)]) = h(t), \qquad \delta = \pm 1$$

by using standard methods.

For related results, we refer to Komkov and Waid [1] and Komkov [2].

Let $I \equiv [t_0, \infty)$ for some fixed $t_0 \geq 0$. It is assumed throughout this note that

(*a*) $h, g_i \in C[I, R]$ and $\lim_{t\to\infty} g_i(t) = \infty$ for $i = 1, \cdots, m$.

(*b*) $f \in C[I \times R^m, R]$.

(*c*) $f(t, y_1, \cdots, y_m)$ be a nondecreasing function with respect to $y_1, \cdots, y_m$ and

$$0 < f(t, y_1, \cdots, y_m) \leq -f(t, -y_1, \cdots, -y_m)$$

for $y_i > 0$ and $i = 1, \cdots, m$.

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2. Main Results

THEOREM 1. *Let*

$$(C_1) \qquad \int^{\infty} h(t)\, dt \quad \textit{converge}$$

$$(C_2) \qquad \int^{\infty} |f(t, c, \cdots, c)|\, dt = \infty$$

for any nonzero constant c. Then every solution of E (1) *cannot be bounded away from zero.*

Proof. Assume to the contrary that there exists a solution $x(t)$ of E (1) such that $x(t)$ is bounded away from zero on I. Without loss of generality, we assume that $x(t) > c > 0$ for some constant c. Condition (*a*) implies that there exists a $t_1 \geq t_0$ such that

$$x[g_i(t)] > c$$

for $t \geq t_1$ and $i = 1, \cdots, m$.

Hence, by (*b*) and (*c*),

$$(1) \qquad f(t, x[g_1(t)], \cdots, x[g_m(t)]) \geq f(t, c, \cdots, c) > 0$$

for $t \geq t_1$. It follows from (C_1) that there is a $T \geq t_1$ such that

$$(2) \qquad \int_T^{\infty} h(t)\, dt < 1 .$$

Integrating E (1) from T to t and using (1) and (2),

$$x^{(n-1)}(t) = x^{(n-1)}(T) + \int_T^t [h(s) - f(s, x[g_1(s)], \cdots, x[g_m(s)])\, ds$$

$$\leq x^{(n-1)}(T) + 1 - \int_T^t f(s, c, \cdots, c)\, ds \to -\infty$$

as $t \to \infty$, which implies $x(t) \to \infty$ as $t \to \infty$, a contradiction. Thus our proof is complete.

COROLLARY. *Under the assumptions of Theorem 1, every solution of* E (1) *is oscillatory or such that* $\liminf_{t \to \infty} |x(t)| = 0$.

Example 1. The differential equation

$$x''(t) + 4^{-1} t^{-2} x(t) = 0$$

does not satisfy condition (C_2). This equation has a nonoscillatory solution $x(t) = t^{\frac{1}{2}}$ which is bounded away from zero.

Example 2. Consider the equation

$$x''(t) + t\,[x(t)]^3 = 2\,t^{-3} + t^{-2}. \tag{3}$$

We see easily that the conditions of Theorem 1 are satisfied. Hence, every solution $x(t)$ of (3) is oscillatory or such that $\liminf_{t\to\infty} |x(t)| = 0$. In fact, $x(t) = t^{-1}$ is a nonoscillatory solution of (3) having this property.

THEOREM 2. *Let*

$$(C_3) \qquad h(t) > 0 \quad \text{and} \quad \int^{\infty} h(t)\,dt = \infty$$

$$(C_4) \qquad \int^{\infty} f(t, c, \cdots, c)\,dt \quad \text{converge}$$

for any nonzero constant c. Then every solution of E(δ) *is unbounded.*

Proof. We only consider the case E(1). Assume to the contrary that there exists a solution $x(t)$ of E(1) which is bounded. Then there exist $T \geq t_0$ and $c > 0$ such that for $t \geq T$

$$-c \leq x[g_i(t)] \leq c$$

which, by condition (c), implies

$$f(t, -c, \cdots, -c) \leq f(t, x[g_1(t)], \cdots, x[g_m(t)]) \leq f(t, c, \cdots, c).$$

Integrating E(1) from T to t and using condition (C_3) and (C_4)

$$x^{(n-1)}(t) = x^{(n-1)}(T) + \int_T^t h(s)\,ds - \int_T^t f(s, x[g_1(s)], \cdots, x[g_m(s)])\,ds$$

$$\geq x^{(n-1)}(T) + \int_T^t h(s)\,ds - \int_T^t f(s, c, \cdots, c)\,ds \to \infty$$

as $t \to \infty$, i.e. $x(t) \to \infty$, a contradiction. Thus the proof is complete.

THEOREM 3. *Under the condition* (C_1), *moreover, assume that*

$$f(t, y_1, \cdots, y_m) = p(t)\,F(y_1, \cdots, y_m)$$

where $p(t) \in C[I, R]$ *and* $p(t) > 0$. *If*

$$(C_5) \qquad \int^{\infty} p(t)\,dt = \infty$$

and

$$(C_6) \qquad \liminf_{t\to\infty} \frac{\int^t h(s)\,ds}{\int^t p(s)\,ds} \geq k > 0\,,$$

then no positive (negative) nonoscillatory solution of E (1) (E (— 1)) *approaches zero.*

Proof. We only prove the case E (1). Let $x(t)$ be a positive nonoscillatory solution of E (1), which approaches zero. Then there exists a $T \geq t_0$ such that for all $t \geq T$

$$(3) \qquad F(x[g_1(t)], \cdots, x[g_m(t)]) < 4^{-1} k\,.$$

Integrating E (1) from T to t

$$x^{(n-1)}(t) - x^{(n-1)}(T) = -\int_T^t p(s)\, F(x[g_1(s)], \cdots, x[g_m(s)])\,ds + \int_T^t h(s)\,ds$$

This and (3) imply

$$x^{(n-1)}(t) - x^{(n-1)}(T) \geq -4^{-1} k \int_T^t p(s)\,ds + \int_T^t h(s)\,ds$$

which, by (C_6), yields

$$(4) \qquad \frac{x^{(n-1)}(t)}{\int_T^t p(s)\,ds} - \frac{x^{(n-1)}(T)}{\int_T^t p(s)\,ds} \geq -4^{-1} k + \frac{\int_T^t h(s)\,ds}{\int_T^t p(s)\,ds} \geq -4^{-1} k + 2^{-1} k$$

$$= 2^{-1} k > 0$$

as $t \to \infty$.

It follows from (C_5) and (4) that $x^{(n-1)}(t) \to \infty$ as $t \to \infty$, which in term forces $x(t) \to \infty$ as $t \to \infty$, a contradiction. Thus our proof is complete.

References

[1] V. Komkov and C. Waid (1973) – *Asymptotics behavior of nonlinear inhomogeous equations via nonstandard analysis*, « Ann. Polonici Math. », *28*, 67–87.

[2] V. Komkov (1974) – *Asymptotics behavior of nonlinear inhomogeous differential equations via nonstandard analysis*, « Ann. Polonici Math. », *30*, 205–219.

[3] H. Murakami, S. Nakagiri and C. C. Yeh – *Asymptotic behavior of solutions of nonlinear differential equations with deviating arguments via nonstandard analysis*, to appear.