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Takashi Noiri

On pairwise s-regular spaces

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Topologia. — *On pairwise s-regular spaces.* Nota di TAKASHI NOIRI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — In questa Nota si ottengono proprietà degli spazi di cui nel titolo, recentemente introdotti da Maheshwari e Prasad [3].

1. INTRODUCTION

In 1963, J. C. Kelly [1] has defined a bitopological space (X, Γ_1, Γ_2) to be pairwise regular if for each point $x \in X$ and each Γ_i-closed set F not containing x, there exist a Γ_i-open set U and a Γ_j-open set V such that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$, where $i \neq j$, $i, j = 1, 2$. On the other hand, N. Levine [2] introduced the concept of semi-open sets in a topological space. Recently, by using the semi-open set S. N. Maheshwari and R. Prasad [3] have introduced the concept of pairwise s-regularity as a generalization of pairwise regularity. The purpose of the present note is to give some properties of pairwise s-regularity.

Let S be a subset of a topological space (X, Γ). We will denote the closure of S and the interior of S by $Cl_X(S)$ and $Int_X(S)$, respectively. The subset S is said to be *semi-open* [2] if there exists an open set U such that $U \subset S \subset Cl_X(U)$. The family of all semi-open sets of (X, Γ) will be denoted by $SO(X, \Gamma)$ (simply, $SO(\Gamma)$). The complement of a semi-open set is said to be *semi-closed*. The intersection of all semi-closed sets containing the subset S is called the *semi-closure* of S and is denoted by $sCl(S)$.

2. PAIRWISE s-REGULAR SPACES

DEFINITION 1. A bitopological space (X, Γ_1, Γ_2) is said to be *pairwise s-regular* [3] if for each Γ_i-closed set F and each point $x \notin F$, there exist a Γ_j-semi-open set U and a Γ_i-semi-open set V such that $F \subset U$, $x \in V$ and $U \cap V = \emptyset$, where $i \neq j$, $i, j = 1, 2$.

DEFINITION 2. A subset S of a topological space (X, Γ) is called an *α-set* [4] if $S \subset Int_X(Cl_X(Int_X(S)))$.

Remark 1. Every open set is an α-set and every α-set is semi-open.

LEMMA 1 (Njåstad [4]). *If* A *is an α-set of a topological space* (X, Γ) *and* $U \in SO(X, \Gamma)$, *then* $U \cap A \in SO(X, \Gamma)$.

Remark 2. In [4], O. Njåstad called semi-open sets β-sets.

(*) Nella seduta del 23 giugno 1977.

THEOREM 1. *If* (X, Γ_1, Γ_2) *is a pairwise s-regualr space and* A *is a bi-α-set of* (X, Γ_1, Γ_2), *then the subspace* $(A, \Gamma_1/A, \Gamma_2/A)$ *is pairwise s-regular.*

Proof. Let F_A be a Γ_i/A-closed set and $x \in A - F_A$. Then there exists a Γ_i-closed set F such that $F_A = F \cap A$. Since (X, Γ_1, Γ_2) is pairwise s-regular and $x \notin F$, there exist $U \in SO(\Gamma_j)$ and $V \in SO(\Gamma_i)$ such that $F \subset U$, $x \in V$ and $U \cap V = \varnothing$. Now put $U_A = U \cap A$ and $V_A = V \cap A$, then $F_A \subset U_A$, $x \in V_A$ and $U_A \cap V_A = \varnothing$. Since A is a bi-α-set, by Lemma 1 we have $U_A \in SO(\Gamma_j)$ and $V_A \in SO(\Gamma_i)$. Moreover, it follows from Theorem 6 of [2] that $U_A \in SO(\Gamma_j/A)$ and $V_A \in SO(\Gamma_i/A)$. This shows that $(A, \Gamma_1/A, \Gamma_2/A)$ is pairwise s-regular.

COROLLARY. *Every bi-open subspace of a pairwise s-regular space is pairwise s-regular.*

We shall give a characterization of pairwise s-regular spaces.

THEOREM 2. *A bitopological space* (X, Γ_1, Γ_2) *is pairwise s-regular if and only if for each point* $x \in X$ *and each* Γ_i*-open set* V *containing* x, *there exists a* Γ_i*-semi-open set* U *such that* $x \in U \subset \Gamma_j - s\,\mathrm{Cl}(U) \subset V$, *where* $i \neq j$, $i, j = 1, 2$.

Proof. Necessity. Let $x \in X$ and V be a Γ_i-open set containing x. Then $X - V$ is a Γ_i-closed set not containing x. Since (X, Γ_1, Γ_2) is pairwise s-regular, there exist $W \in SO(\Gamma_j)$ and $U \in SO(\Gamma_i)$ such that $X - V \subset W$, $x \in U$ and $U \cap W = \varnothing$. Therefore, we obtain

$$x \in U \subset \Gamma_j - s\,\mathrm{Cl}(U) \subset \Gamma_j - s\,\mathrm{Cl}(X - W) = X - W \subset V.$$

Sufficiency. Let F be a Γ_i-closed set not containing x. Then, there exists a $V \in SO(\Gamma_i)$ such that $x \in V \subset \Gamma_j - s\,\mathrm{Cl}(V) \subset X - F$. Put $U = X - \Gamma_j - s\,\mathrm{Cl}(V)$, then we have $F \subset U \in SO(\Gamma_j)$ and $U \cap V = \varnothing$. This shows that (X, Γ_1, Γ_2) is pairwise s-regular.

Let $\{(X_\alpha, \Gamma_\alpha, \Gamma_\alpha^*) \mid \alpha \in \nabla\}$ be a family of bitopological spaces. By X, Γ and Γ^* we shall denote the product set $\prod_{\alpha \in \nabla} X_\alpha$, the product topologies $\prod_{\alpha \in \nabla} \Gamma_\alpha$ and $\prod_{\alpha \in \nabla} \Gamma_\alpha^*$, respectively.

LEMMA 2 (Noiri [5]). *Let* $\{(X_\alpha, \Gamma_\alpha) \mid \alpha \in \nabla\}$ *be any family of topological spaces, where* ∇ *is non-empty. If* S_α *is a non-empty subset of* X_α *for each* $\alpha \in \nabla$, *then*

$$\Gamma - s\,\mathrm{Cl}\left(\prod_{\alpha \in \nabla} S_\alpha\right) \subset \prod_{\alpha \in \nabla} \Gamma_\alpha - s\,\mathrm{Cl}(S_\alpha).$$

THEOREM 3. *If* $(X_\alpha, \Gamma_\alpha, \Gamma_\alpha^*)$ *is pairwise s-regular for each* $\alpha \in \nabla$, *then* (X, Γ, Γ^*) *is pairwise s-regular.*

Proof. Let $x = (x_\alpha) \in X$ and V be a Γ-open set containing x. Then there exists a basic open set ΠV_α such that $x \in \Pi V_\alpha \subset V$, where V_α is

Γ_α-open for each $\alpha \in \nabla$ and $V_\alpha = X_\alpha$ for all except for a finite number of ∇, say $\alpha_1, \alpha_2, \cdots, \alpha_n$. Since $(X_\alpha, \Gamma_\alpha, \Gamma^*_\alpha)$ is pairwise s-regular and $x_\alpha \in V_\alpha \in \Gamma_\alpha$ for each $\alpha \in \nabla$, by Theorem 2, there exists a $U_{\alpha_j} \in SO(\Gamma_{\alpha_j})$ such that

$$x_{\alpha_j} \in U_{\alpha_j} \subset \Gamma^*_{\alpha_j} - s\,Cl(U_{\alpha_j}) \subset V_{\alpha_j},$$

where $j = 1, 2, \cdots, n$. Let us put

$$U = \prod_{j=1}^{n} U_{\alpha_j} \times \prod_{\alpha \neq \alpha_j} X_\alpha,$$

then $U \in SO(\Gamma)$ [6, Theorem 2]. Moreover, by Lemma 2, we have

$$x \in U \subset \Gamma^* - sCl(U) \subset \prod_{j=1}^{n} \Gamma^*_{\alpha_j} - s\,Cl(U_{\alpha j}) \times \prod_{\alpha \neq \alpha_j} X_\alpha \subset V.$$

In the case that V is Γ^*-open, the above facts are established quite similarly. Therefore, it follows from Theorem 2 that (X, Γ, Γ^*) is pairwise s-regular.

DEFINITION 3. A function $f: (X, \Gamma) \to (Y, \Gamma^*)$ is said to be *semi-closed* [7] if the image $f(F)$ of each closed set F of (X, Γ) is semi-closed in (Y, Γ^*).

Remark 3. Every closed function is semi-closed, but the converse does not hold [7, Example].

THEOREM 4. *Let f be a pairwise continuous and pairwise semi-closed function of a bitopological space (X, Γ_1, Γ_2) onto a bitopological space $(Y, \Gamma^*_1, \Gamma^*_2)$, that is, $f: (X, \Gamma_1) \to Y, \Gamma^*_1)$ and $f: (X, \Gamma_2) \to (Y, \Gamma^*_2)$ both are continuous and semi-closed. If (X, Γ_1, Γ_2) is pairwise regular and $f^{-1}(y)$ is Γ_1-compact and Γ_2-compact for each $y \in Y$, then $(Y, \Gamma^*_1, \Gamma^*_2)$ is pairwise s-regular.*

Proof. Let F^* be a Γ^*_i-closed set and $y \in Y - F^*$. Then $f^{-1}(F^*)$ is Γ_i-closed and $f^{-1}(F^*) \cap f^{-1}(y) = \varnothing$. Since (X, Γ_1, Γ_2) is pairwise regular, for each $x \in f^{-1}(y)$ there exists a Γ_j-open set U_x and a Γ_i-open set V_x such that $f^{-1}(F^*) \subset U_x$, $x \in V_x$ and $U_x \cap V_x = \varnothing$. The family $\{V_x \mid x \in f^{-1}(y)\}$ is a Γ_i-open cover of $f^{-1}(y)$. Since $f^{-1}(y)$ is Γ_i-compact, there exists a finite number of points $x_1, x_2, \cdots, x_n$ in $f^{-1}(y)$ such that $f^{-1}(y) \subset \cup \{V_{x_j} \mid j = 1, 2, \cdots, n\}$. Now put

$$V = \bigcup_{j=1}^{n} V_{x_j} \quad \text{and} \quad U = \bigcap_{j=1}^{n} U_{x_j},$$

then V is Γ_i-open and U is Γ_j-open. Moreover, we have $f^{-1}(y) \subset V$, $f^{-1}(F^*) \subset U$ and $U \cap V = \varnothing$. Since f is pairwise semi-closed, there exist $V^* \in SO(\Gamma^*_i)$ and $U^* \in SO(\Gamma^*_j)$ such that $y \in V^*$, $F^* \subset U^*$, $f^{-1}(V^*) \subset V$ and $f^{-1}(U^*) \subset U$ [7, Theorem 5]. Since U and V are disjoint, we have $U^* \cap V^* = \varnothing$. This shows that $(Y, \Gamma^*_1, \Gamma^*_2)$ is pairwise s-regular.

REFERENCES

[1] J. C. KELLY (1963) – *Bitopological spaces*, « Proc. London Math. Soc. », (3) *13*, 71–89.

[2] N. LEVINE (1963) – *Semi-open sets and semi-continuity in topological spaces*, « Amer. Math. Monthly », *70*, 36–41.

[3] S. N. MAHESHWARI and R. PRASAD – *On pairwise s-regular spaces* (under communication).

[4] O. NJÅSTAD (1965) – *On some classes of nearly open sets*, « Pacific J. Math. », *15*, 961–970.

[5] T. NOIRI – *A note on s-regular spaces* (to appear.)

[6] T. NOIRI (1973) – *On semi-continuous mappings*, « Atti Accad. Naz. Lincei, Rend. Cl. Sci. fis. mat. nat. », (8) *54*, 210–214.

[7] T. NOIRI (1973) – *A generalization of closed mappings*, « Atti Accad. Naz. Lincei, Rend. Cl. Sci fis. mat. nat. », (8) *54*, 412–415.