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Geometria differenziale. — *Submanifolds of a manifold with areal metric*, I. Nota di SURESH CHANDRA RASTOGI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Nel 1966 H. Rund [4] ha esaminato la possibilità di introdurre un tensore metrico a quattro indici in luogo di quello usuale a due indici; nel 1968 egli [5] ha poi definiti spazi areolari dipendenti da un intero l che, rispettivamente per $l = 1$ ed $l = n - 1$, si riducono a spazi di Finsler e di Cartan. L'Autore ha quindi studiato [2, 3] le sottovarietà di quegli spazi introducendo per esse dei coefficienti di connessione. Nel presente lavoro si sviluppa la teoria della curvatura indotta su di una sottovarietà di una varietà a metrica areolare, e si definisce la derivata di Lie di un campo di vettori facendone applicazioni alle suddette sottovarietà.

1. INTRODUCTION

Let X_n be a differentiable manifold referred to local coordinates $x^i = x^i(t^\alpha)$; $(i = 1, \cdots, n ; \alpha = 1, \cdots, l)$, t^α denoting a system of independent parameters of an l-dimensional subspace C_l of X_n $(l < n)$; then we have $\dot{x}^i_\alpha = \partial x^i/\partial t^\alpha$.

Let $L(x^i, \dot{x}^i_\alpha)$ be a function of $n + nl$ variables $x^i, \dot{x}^i_\alpha$ satisfying all the conditions of Rund [5]; then, for $\dot{\partial}^\alpha_i = \partial/\partial\dot{x}^i_\alpha$, the metric tensor of X_n is defined by

$$g^{\alpha\beta}_{ij}(x^h, \dot{x}^h_\varepsilon) = \frac{l}{2}\, \dot{\partial}^\alpha_i\, \dot{\partial}^\beta_j \{L(x^h, \dot{x}^h_\varepsilon)\}^{2/l} . \tag{1.1}$$

Let u^A $(A = 1, \cdots, m)$ and $g^{\alpha\beta}_{AB}(u^D, \dot{u}^D_\varepsilon)$ be the coordinates and metric tensor of X_m, imbedded in X_n and parameterized by C_l; then we have [2]:

$$g^{\alpha\gamma}_{AC}(u^D, \dot{u}^D_\varepsilon) = g^{\alpha\gamma}_{ik}(x^h, \dot{x}^h_\varepsilon)\, B^i_A\, B^k_C , \tag{1.2}$$

where $B^i_A = \partial_A x^i$, $(\partial_A = \partial/\partial u^A)$.

Let $\underset{\mu}{N}{}^\alpha_i$ $(\mu = m + 1, \cdots, n)$ be the $l(n-m)$ linearly independent unit vectors normal to X_m; then they satisfy [2]:

$$\underset{\mu}{N}{}^i_\alpha = g^{ij}_{\alpha\beta}(x^h, \dot{x}^h_\varepsilon)\, \underset{\mu}{N}{}^\alpha_j , \tag{1.3}$$

$$g^{\alpha\beta}_{ij}(x^h, \dot{x}^h_\varepsilon)\, \underset{\mu}{N}{}^i_\alpha\, \underset{\nu}{N}{}^j_\beta = \delta^\mu_\nu , \tag{1.4}$$

$$\underset{\mu}{N}{}^\alpha_i\, B^i_A = 0 \quad , \quad B^A_i\, \underset{\mu}{N}{}^i_\alpha = 0 , \tag{1.5}$$

where $B^A_i = g^{AD}_{\alpha\delta}\, g^{\alpha\delta}_{il}\, B^l_D$.

(*) Nella seduta del 23 giugno 1977.

We have also

(1.6) $$B^A_i \, B^i_C = \delta^A_C \, ,$$

(1.7) $$B^i_A \, B^A_j = \delta^i_j - N^i_j (x^h, \dot{x}^h_\varepsilon) \, ,$$

where

(1.8) $$N^i_j (x^h, \dot{x}^h_\varepsilon) \overset{\text{def}}{=} \sum_{\mu=m+1}^{n} \underset{\mu}{N}^i_\alpha \underset{\mu}{N}^\alpha_j \, .$$

The covariant differential of a vector field X^i_ε is given by [3]:

(1.9) $$DX^i_\varepsilon = X^i_{\varepsilon,h} \, dx^h + X^{i}_{\varepsilon,}{}^{\gamma}_{,h} \, Dm^h_\gamma \, ,$$

where

(1.10) $$X^i_{\varepsilon,h} = \partial_h X^i_\varepsilon - (\dot{\partial}^\gamma_k X^i_\varepsilon) M^{*k}_{rh} \dot{x}^r_\gamma + X^l_\varepsilon M^{*i}_{lh}$$

and

(1.11) $$X^{i}_{\varepsilon,}{}^{\gamma}_{,h} = L^{1/l} \, l^{1/2} \, \dot{\partial}^\gamma_h X^i_\varepsilon + X^l_\varepsilon A^{i\gamma}_{lh} \, .$$

The terms used in (1.10) and (1.11) are given by

(1.12) $$M^{*i}_{kj} = M^i_{kj} - C^{i\alpha}_{kl} \, M^l_{rj} \, \dot{x}^r_\alpha \, ,$$

(1.13) $$M^k_{ih} = g^{ki}_{\alpha\beta} \, M^{\alpha\beta}_{ijh} \, ,$$

(1.14) $$M^{\alpha\beta}_{ijh} + M^{\beta\alpha}_{ijh} = \partial_h g^{\alpha\beta}_{ij} \, ,$$

(1.15) $$A^{i\gamma}_{lh} = L^{1/l} \, l^{1/2} \, C^{i\gamma}_{lh} \, ,$$

where

(1.16) $$C^{k\gamma}_{ih} = C^{\alpha\beta\gamma}_{ijh} \, g^{kj}_{\alpha\beta}$$

and

(1.17) $$2 \, C^{\alpha\beta\gamma}_{ijh} = \dot{\partial}^\gamma_k g^{\alpha\beta}_{ij} \, .$$

Here m^h_γ represents a unit vector in X_n equivalent to $L^{-1/l} \, l^{-1/2} \, \dot{x}^h_\gamma$.

2. INDUCED CONNECTION COEFFICIENTS

Let $\bar{D}X^A_\varepsilon$ be the covariant differential of X^A_ε with respect to X_m; then we have

(2.1) $$\bar{D}X^A_\varepsilon = B^A_i \, DX^i_\varepsilon \, ,$$

where

(2.2) $$DX^i_\varepsilon = dX^i_\varepsilon + C^{i\gamma}_{kl} \, X^k_\varepsilon \, d\dot{x}^l_\gamma + M^i_{kl} \, X^k_\varepsilon \, dx^l \, .$$

Let dx^k and du^C be the corresponding displacements in X_n and X_m respectively and $B^i_{AC} = \partial_C B^i_A$; then on differentiating $X^i_\varepsilon = B^i_A X^A_\varepsilon$, we obtain

$$dX^i_\varepsilon = (dX^A_\varepsilon) B^i_A + B^i_{AC} X^A_\varepsilon du^C . \tag{2.3}$$

Using a similar relation for the corresponding change in $d\dot{x}^k_\gamma$ of (2.2) and applying (2.1) in the resulting equation we get

$$\bar{D}X^A_\varepsilon = dX^A_\varepsilon + C^{A\alpha}_{ED} X^E_\varepsilon d\dot{u}^D_\alpha + M^A_{CD} X^C_\varepsilon du^D , \tag{2.4}$$

where

$$M^A_{CD} \overset{\text{def}}{=} (B^i_{CD} + C^{i\alpha}_{kh} B^h_C B^k_{ED} \dot{u}^E_\alpha + M^i_{hk} B^{hk}_{CD}) B^A_i . \tag{2.5}$$

Similarly for another covariant differential defined by

$$D^* X^i_\varepsilon = dX^i_\varepsilon + M^{*i}_{jk} X^j_\varepsilon dx^k , \tag{2.6}$$

we have

$$M^{*A}_{CD} = B^A_i (B^i_{CD} + M^{*i}_{hk} B^{hk}_{CD}) \tag{2.7}$$

such that

$$M^A_{BC} \dot{u}^B_\alpha = M^{*A}_{BC} \dot{u}^B_\alpha .$$

Let Y^i_α be a vector attached to X_n and normal to X_m; then $Y^i_\alpha = \sum_\mu Y^\mu \underset{\mu}{N}{}^i_\alpha$, ($\mu = m+1, \cdots, n$). Since the covariant differential $\bar{D}Y^\mu$ of Y^μ is defined by

$$\bar{D}Y^\mu = g^{\alpha\beta}_{ij} \underset{\mu}{N}{}^j_\beta DY^i_\alpha , \tag{2.8}$$

writing

$$\bar{D}Y^\mu = dY^\mu + \bar{C}^{\mu\alpha}_{\nu A} Y^\nu d\dot{u}^A_\alpha + \bar{\lambda}^\mu_{\nu A} Y^\nu du^A \tag{2.9}$$

and using (2.8) we obtain

$$\bar{C}^{\mu\varepsilon}_{\nu A} = g^{\alpha\beta}_{ij} \underset{\mu}{N}{}^i_\alpha \underset{\nu}{N}{}^h_\beta B^k_A C^{j\varepsilon}_{hk} \tag{2.10}$$

and

$$\bar{\lambda}^\mu_{\nu A} = g^{\alpha\beta}_{ij} \underset{\mu}{N}{}^j_\beta (\partial_A \underset{\nu}{N}{}^i_\alpha + M^i_{hk} B^k_A \underset{\nu}{N}{}^h_\alpha + C^{i\delta}_{hk} \underset{\nu}{N}{}^h_\alpha B^k_{DA} \dot{u}^D_\delta) , \tag{2.11}$$

where we have assumed

$$g^{\alpha\beta}_{ij} \underset{\mu}{N}{}^i_\alpha \dot{\partial}^\varepsilon_A \underset{\nu}{N}{}^j_\beta = 0 .$$

Putting

$$\lambda^\mu_{\nu A} \overset{\text{def}}{=} \bar{\lambda}^\mu_{\nu A} - \bar{C}^{\mu\alpha}_{\nu B} M^B_{DA} \dot{u}^D_\alpha ,$$

in (2.9) we get

$$\bar{D}Y^\mu = dY^\mu + \bar{A}^{\mu\alpha}_{\nu B} Y^\nu \bar{D}m^B_\alpha + \lambda^\mu_{\nu A} Y^\nu du^A , \tag{2.12}$$

where $\bar{A}^{\mu\alpha}_{\nu B}$ is a term in X_m corresponding to $A^{i\alpha}_{hk}$ in X_n.

3. NORMAL CURVATURE

Consider a curve $C : x^i = x^i(s)$ of X_m referred to its arc length as parameter. We define the normal curvature vector on X_m at a point P of the curve C in the direction x'^i_α (dashes denoting differentiation with respect to the arc length of the curve C), tangent to the curve C, by the vector $D x'^i_\alpha - B^i_A \bar{D} u'^A_\alpha$, normal to X_m.

One can easily observe that

$$(3.1) \qquad D m^i_\alpha - B^i_A \bar{D} m^A_\alpha = H^i_{\alpha A}\, du^A ,$$

where

$$(3.2) \qquad H^i_{\alpha A} \overset{\text{def}}{=} (B^i_{CA} - B^i_D M^D_{CA})\, m^C_\alpha + B^k_A M^i_{hk}\, m^h_\alpha .$$

Using the covariant differentials (2.2) and (2.4) in terms of the covariant differential $\bar{D} m^A_\alpha$, we have

$$(3.3) \qquad D X^i_\alpha - B^i_A \bar{D} X^A_\alpha = (L^{1/l}\, l^{1/2}\, \dot{\partial}^\varepsilon_h X^i_\alpha + A^{i\varepsilon}_{hk} X^k_\alpha)\, D m^h_\varepsilon + X^i_{\alpha,h}\, dx^h$$
$$- (L^{1/l}\, l^{1/2}\, \dot{\partial}^\beta_B X^D_\alpha + A^{D\beta}_{CB} X^C_\alpha)\, \bar{D} m^B_\beta\, B^i_D - B^i_D X^D_{\alpha,C}\, du^C,$$

where

$$(3.4) \qquad X^A_{\alpha,C} \overset{\text{def}}{=} \partial_C X^A_\alpha - (\dot{\partial}^\varepsilon_E X^A_\alpha)\, M^E_{DC}\, \dot{u}^D_\varepsilon + M^{*A}_{DC} X^D_\alpha .$$

Solving the right hand side of equation (3.3) term by term and making use of

$$\overline{M}^{*B}_{DC} = M^B_{DC} - A^{B\alpha}_{DE} M^E_{AC}\, m^A_\alpha ,$$

we obtain on simplification

$$(3.6) \qquad D X^i_\alpha - B^i_A \bar{D} X^A_\alpha = (B^i_{DC} - B^i_A \overline{M}^{*A}_{DC} + M^{*i}_{hk} B^{hk}_{DC} + A^{i\beta}_{hk} B^h_D H^k_{\beta C})\, X^D_\alpha\, du^C$$
$$+ N^i_j A^{j\varepsilon}_{hk} B^{hk}_{DC} X^D_\alpha \bar{D} m^C_\varepsilon .$$

From equation (3.6) for a vector tangent to the curve C, we can obtain

$$(3.7) \qquad D x'^i_\alpha - B^i_A \bar{D} u'^A_\alpha = \overset{\circ}{H}{}^i_{BC}\, u'^B_\alpha\, du^C ,$$

where

$$(3.8) \qquad \overset{\circ}{H}{}^i_{CD} \overset{\text{def}}{=} B^i_{DC} - B^i_A \overline{M}^{*A}_{DC} + (M^{*i}_{hk} B^k_C + A^{i\alpha}_{hk} H^k_{\alpha C})\, B^h_D ,$$

is called the normal curvature tensor at the point considered in the direction of u'^A_α.

Multiplying equation (3.8) by m^D_α we can easily obtain

$$\overset{\circ}{H}{}^i_{CD}\, m^D_\alpha = H^i_{\alpha C} .$$

By defining

$$(3.9) \qquad M^i_{hA} \overset{\text{def}}{=} M^{*i}_{hk} B^k_A + A^{i\alpha}_{hk} H^k_{\alpha A} ,$$

we can write (3.8) as

$$\overset{\circ}{H}{}^{i}_{CD} = B^{i}_{DC} + M^{i}_{hC}\, B^{h}_{D} - M^{*A}_{DC}\, B^{i}_{A}\,, \tag{3.10}$$

which represents a mixed covariant derivative of B^i_D.

Defining

$$\overset{\circ}{D}_{A}\, X^{i}_{\alpha} \overset{\text{def}}{=} B^{h}_{A}\, X^{i}_{\alpha,h} + H^{h}_{\beta A}\, X^{i}_{\alpha,,h}{}^{\beta} \tag{3.11}$$

and

$$\overset{|}{D}{}^{\gamma}_{A}\, X^{i}_{\alpha} \overset{\text{def}}{=} B^{h}_{A}\, X^{i}_{\alpha,,h}{}^{\gamma}\,, \tag{3.12}$$

equation (1.9) can be expressed as

$$DX^{i}_{\alpha} = \overset{\circ}{D}_{A}\, X^{i}_{\alpha}\, du^{A} + \overset{|}{D}{}^{\gamma}_{A}\, X^{i}_{\alpha}\, \overline{D}m^{A}_{\gamma}\,. \tag{3.13}$$

The $\overset{\circ}{D}$–differentiation of X^i_α, X^A_α and Y^μ is expressible as

$$\overset{\circ}{D}_{A}\, X^{i}_{\alpha} \overset{\text{def}}{=} \partial_{A}\, X^{i}_{\alpha} - L^{1/l}\, l^{1/2}\, (\dot\partial^{\beta}_{B}\, X^{i}_{\alpha})\, M^{B}_{CA}\, m^{C}_{\beta} + M^{i}_{kA}\, X^{k}_{\alpha}\,, \tag{3.14}$$

$$\overset{\circ}{D}_{B}\, X^{A}_{\alpha} \overset{\text{def}}{=} \partial_{B}\, X^{A}_{\alpha} - (\dot\partial^{\gamma}_{C}\, X^{A}_{\alpha})\, M^{C}_{DB}\, \dot{u}^{D}_{\gamma} + \overline{M}^{*A}_{BC}\, X^{C}_{\gamma} \tag{3.15}$$

and

$$\overset{\circ}{D}_{A}\, Y^{\mu} \overset{\text{def}}{=} \partial_{A}\, Y^{\mu} - (\dot\partial^{\gamma}_{B}\, Y^{\mu})\, M^{B}_{DA}\, \dot{u}^{D}_{\gamma} + \lambda^{\mu}_{\nu A}\, Y^{\nu}. \tag{3.16}$$

Using equations (2.5), (2.7) and (3.2) together with

$$A^{E\alpha}_{CD} = B^{E}_{i}\, A^{i\alpha}_{hk}\, B^{hk}_{CD}\,, \tag{3.17}$$

in (3.5) we get on simplification

$$\overline{M}^{*A}_{CD} = M^{*A}_{CD} + B^{A}_{i}\, A^{i\alpha}_{hk}\, B^{h}_{C}\, H^{k}_{\alpha D}\,. \tag{3.18}$$

From equations (3.8) and (3.18) we get

$$\overset{\circ}{H}{}^{i}_{[AC]} = N^{i}_{j}\, A^{j\alpha}_{hk}\, H^{k}_{\alpha[A}\, B^{h}_{C]}\ {}^{(1)}. \tag{3.19}$$

Multiplying equation (3.19) by $\underset{\mu}{N}{}^{\alpha}_{i}$ and using

$$\overset{\circ}{H}{}^{\mu\alpha}_{AC} \overset{\text{def}}{=} \overset{\circ}{H}{}^{i}_{AC}\, \underset{\mu}{N}{}^{\alpha}_{i}\,,$$

we find on simplification

$$\overset{\circ}{H}{}^{\mu\alpha}_{[AC]} = N^{i}_{j}\, \underset{\mu}{N}{}^{\alpha}_{i}\, A^{j\alpha}_{hk}\, H^{k}_{\alpha[A}\, B^{h}_{C]}\,. \tag{3.20}$$

(1) The square and round brackets are used to denote the skew-symmetric and symmetric parts respectively of the object with respect to the indices enclosed therein. The index with a bold face is free from these properties.

Observing that

(3.21) $$\overset{|}{D}{}^{\alpha}_{A}\, X^{C}_{\gamma} = B^{C}_{i}\, B^{h}_{A}\, X^{i}_{\gamma,,h}{}^{\alpha} ,$$

(3.22) $$\overset{|}{D}{}^{\alpha}_{A}\, Y^{\mu} = g^{\alpha\beta}_{ij}\, B^{h}_{A}\, \underset{\mu}{N}{}^{i}_{\gamma}\, Y^{j}_{\beta,,h}{}^{\gamma}$$

and defining

(3.23) $$\overset{|}{D}{}^{\alpha}_{A}\, B^{i}_{C} \overset{\text{def}}{=} \overset{|}{H}{}^{i\alpha}_{AC} = B^{h}_{A}\, B^{j}_{C}\, (B^{i}_{D}\, B^{D}_{j})_{,,h}{}^{\gamma}$$

we can easily get

(3.24) $$\overset{|}{H}{}^{i\alpha}_{[AC]} = N^{i}_{j}\, B^{h}_{A}\, B^{k}_{C}\, A^{j\alpha}_{[hk]} .$$

Multiplying equation (3.24) by $g^{AC}_{\beta\gamma}$ and using

$$\overset{|}{H}{}^{i\alpha}_{\beta\gamma} \overset{\text{def}}{=} \overset{|}{H}{}^{i\alpha}_{AC}\, g^{AC}_{\beta\gamma} \quad \text{and} \quad A^{j\alpha}_{\beta\gamma} \overset{\text{def}}{=} A^{j\alpha}_{hk}\, g^{hk}_{\beta\gamma}$$

we obtain

(3.25) $$\overset{|}{H}{}^{i\alpha}_{[\beta\gamma]} = N^{i}_{j}\, A^{j\alpha}_{[\beta\gamma]} .$$

Now we shall define the second normal curvature tensor of X_m at a point of X_m in a given direction $\dot{u}^{A}_{\alpha}$ as follows:

(3.26) $$\underset{\mu}{\overset{\circ}{L}}{}^{i}_{A\alpha} \overset{\text{def}}{=} \overset{\circ}{D}_{A}\, \underset{\mu}{N}{}^{i}_{\alpha} = \partial_{A}\, \underset{\mu}{N}{}^{i}_{\alpha} - (\dot{\partial}^{\gamma}_{C}\, \underset{\mu}{N}{}^{i}_{\alpha})\, M^{C}_{DA}\, \dot{u}^{D}_{\gamma} - \lambda^{\nu}_{\mu A}\, \underset{\nu}{N}{}^{i}_{\alpha} + M^{i}_{kA}\, \underset{\mu}{N}{}^{k}_{\alpha} .$$

Defining

(3.27) $$\underset{\mu}{\overset{\circ}{L}}{}^{A}_{B\alpha} \overset{\text{def}}{=} B^{A}_{i}\, \underset{\mu}{\overset{\circ}{L}}{}^{i}_{B\alpha}$$

and applying the $\overset{\circ}{D}_{A}$-derivative to the equation (1.5) we obtain, by virtue of (3.10) and (3.26), that the two normal curvature tensor are related as follows:

(3.28) $$\underset{\mu}{\overset{\circ}{L}}{}^{A}_{B\alpha} = - g^{AE}_{\alpha\beta}\, \overset{\circ}{H}{}^{\mu\beta}_{BE} .$$

4. GAUSS–CODAZZI AND KÜHNE EQUATIONS

Applying the $\overset{\circ}{D}_{A}$-derivative to (3.10), interchanging A and C and substracting the second equation from the first we obtain on simplification

(4.1) $$2\, \overset{\circ}{D}_{[A}\, \overset{\circ}{H}{}^{i}_{C]D} = B^{h}_{D}\, R^{i}_{hCA} - B^{i}_{F}\, R^{F}_{DCA} + 2\, \overset{\circ}{H}{}^{i}_{ED}\, \overline{M}{}^{*E}_{[AC]} ,$$

where

(4.2) $$R^{i}_{hBA} \overset{\text{def}}{=} 2\, \{ \partial_{[A}\, M^{i}_{hB]} - (\dot{\partial}^{\delta}_{D}\, M^{i}_{h[B})\, M^{D}_{EA]}\, \dot{u}^{E}_{\delta} + M^{i}_{k[A}\, M^{k}_{hB]} \}$$

and

$$\text{(4.3)}\qquad R^{F}_{CBA} \overset{\text{def}}{=} 2\,\{\partial_{[A}\,\overline{M}^{*F}_{CB]} - (\dot{\partial}^{\delta}_{D}\,\overline{M}^{*F}_{C[B)})\,M^{D}_{\mathbf{E}A]}\,\dot{u}^{E}_{\delta} + \overline{M}^{*F}_{D[A}\,\overline{M}^{*D}_{\mathbf{CB}]}\}\,.$$

Similarly we can also find

$$\text{(4.4)}\qquad 2\,\mathring{D}_{[A}\,\underset{\mu}{\mathring{L}}{}^{i}_{B]\alpha} = \underset{\mu}{N}{}^{k}_{\alpha}\,R^{i}_{kBA} - \underset{\nu}{N}{}^{i}_{\alpha}\,R^{\nu}_{\mu BA} - 2\,\underset{\mu}{\mathring{L}}{}^{i}_{D\alpha}\,\overline{M}^{*D}_{[BA]}$$

$$- L^{1/l}\,l^{1/2}\,(\dot{\partial}^{\delta}_{D}\,\underset{\mu}{N}{}^{i}_{\alpha})\,R^{D}_{EBA}\,m^{E}_{\delta}\,,$$

where

$$\text{(4.5)}\qquad R^{\nu}_{\mu BA} \overset{\text{def}}{=} 2\,\{\partial_{[A}\,\lambda^{\nu}_{\boldsymbol{\mu}B]} - (\dot{\partial}^{\varepsilon}_{E}\,\lambda^{\nu}_{\mu[B)})\,M^{E}_{\mathbf{F}A]}\,\dot{u}^{F}_{\varepsilon} + \lambda^{\nu}_{\sigma[A}\,\lambda^{\sigma}_{\boldsymbol{\mu}B]}\}\,.$$

Differentiating $\mathring{H}^{i}_{AC} \overset{\text{def}}{=} \mathring{H}^{\mu\alpha}_{AC}\,\underset{\mu}{N}{}^{i}_{\alpha}$ and using (3.26) we get

$$\mathring{D}_{B}\,\mathring{H}^{i}_{AC} = B^{i}_{D}\,\underset{\mu}{\mathring{L}}{}^{D}_{B\alpha}\,\mathring{H}^{\mu\alpha}_{AC} + \underset{\mu}{N}{}^{i}_{\alpha}\,\mathring{D}_{B}\,\mathring{H}^{\mu\alpha}_{AC}\,,$$

which when substituted in (4.1) yields

$$\text{(4.6)}\qquad 2\,\{B^{i}_{D}\,\underset{\mu}{\mathring{L}}{}^{D}_{[A\alpha}\,\mathring{H}^{\mu\alpha}_{B]C} + \underset{\mu}{N}{}^{i}_{\alpha}\,\mathring{D}_{[A}\,\mathring{H}^{\mu\alpha}_{B]C}\}$$

$$= B^{h}_{C}\,R^{i}_{hBA} - B^{i}_{E}\,R^{E}_{CBA} + 2\,\mathring{H}^{i}_{DC}\,B^{D}_{j}\,A^{j\alpha}_{hk}\,B^{h}_{[A}\,H^{k}_{\alpha B]}\,.$$

Multiplying equation (4.6) by B^{E}_{i} and $\underset{\nu}{N}{}^{\alpha}_{i}$ respectively we get on simplification

$$\text{(4.7)}\qquad 2\,\mathring{L}^{E}_{[A\alpha}\,\mathring{H}^{\mu\alpha}_{B]C} = R^{i}_{hBA}\,B^{h}_{C}\,B^{E}_{i} - R^{E}_{CBA}$$

and

$$\text{(4.8)}\qquad \mathring{D}_{[A}\,\mathring{H}^{\mu\alpha}_{B]C} = \mathring{H}^{i}_{DC}\,B^{D}_{j}\,A^{j\alpha}_{hk}\,H^{k}_{\beta[B}\,B^{h}_{A]}\,\underset{\nu}{N}{}^{\alpha}_{i}\,,$$

which are the equations of Gauss and Codazzi respectively.

Again applying $\mathring{D}$-operator to $\underset{\mu}{\mathring{L}}{}^{i}_{A\alpha} \overset{\text{def}}{=} \underset{\mu}{\mathring{L}}{}^{C}_{A\alpha}\,B^{i}_{C}$ and using (3.10) we obtain

$$\mathring{D}_{B}\,\underset{\mu}{\mathring{L}}{}^{i}_{C\alpha} = \underset{\nu}{N}{}^{i}_{\beta}\,\mathring{H}^{\nu\beta}_{BD}\,\underset{\mu}{\mathring{L}}{}^{D}_{C\alpha} + B^{i}_{D}\,\mathring{D}_{B}\,\underset{\mu}{\mathring{L}}{}^{D}_{C\alpha}\,,$$

which when substituted in (4.4) gives

$$\text{(4.9)}\qquad 2\,\{B^{i}_{D}\,\mathring{D}_{[A}\,\underset{\mu}{\mathring{L}}{}^{D}_{B]\alpha} + \underset{\nu}{N}{}^{i}_{\beta}\,\mathring{H}^{\nu\beta}_{[A\mathbf{D}}\,\underset{\mu}{\mathring{L}}{}^{D}_{B]\alpha} + \underset{\mu}{\mathring{L}}{}^{i}_{D\alpha}\,\overline{M}^{*D}_{[BA]}\}$$

$$= \underset{\mu}{N}{}^{k}_{\alpha}\,R^{i}_{kBA} - \underset{\nu}{N}{}^{i}_{\alpha}\,R^{\nu}_{\mu BA} - L^{1/l}\,l^{1/2}\,(\dot{\partial}^{\delta}_{D}\,\underset{\mu}{N}{}^{i}_{\alpha})\,R^{D}_{EBA}\,m^{E}_{\delta}\,.$$

Multiplying equation (4.9) by B^{E}_{i} and $\underset{\nu}{N}{}^{\alpha}_{i}$ respectively we get on simplification

$$\text{(4.10)}\qquad 2\,\{\mathring{D}_{[A}\,\underset{\mu}{\mathring{L}}{}^{E}_{B]\alpha} + B^{E}_{i}\,\underset{\mu}{\mathring{L}}{}^{i}_{D\beta}\,B^{D}_{j}\,A^{j\beta}_{hk}\,H^{k}_{\alpha[A}\,B^{h}_{B]}\}$$

$$= R^{i}_{kBA}\,\underset{\mu}{N}{}^{k}_{\alpha}\,B^{E}_{i} - L^{1/l}\,l^{1/2}\,(\dot{\partial}^{\delta}_{D}\,\underset{\mu}{N}{}^{i}_{\alpha})\,R^{D}_{CBA}\,m^{C}_{\delta}$$

and

$$2\, \overset{\circ}{\mathrm{H}}{}^{\nu\alpha}_{[\mathrm{AD}} \underset{\mu}{\overset{\circ}{\mathrm{L}}}{}^{\mathrm{D}}_{\mathrm{B}]\alpha} = \mathrm{R}^{i}_{k\mathrm{BA}} \underset{\mu}{\mathrm{N}}{}^{k}_{\alpha} \underset{\nu}{\mathrm{N}}{}^{\alpha}_{i} - \mathrm{R}^{\nu}_{\mu\mathrm{BA}}\,, \tag{4.11}$$

which are the equations of Codazzi and Kühne respectively.

5. LIE-DIFFERENTIATION

Let $v^i(x^h, \dot{x}^h_\alpha)$ be a vector field of class C^2 defined over a region R of X_n such that with it we can associate an infinitesimal transformation of the type

$$\bar{x}^i = x^i + v^i(x^h, \dot{x}^h_\alpha)\, d\tau\,, \tag{5.1}$$

where $d\tau$ is an infinitesimal constant.

From equation (5.1) we obtain

$$\dot{\bar{x}}^i_\alpha = \dot{x}^i_\alpha + \{(\partial_h v^i)\, \dot{x}^h_\alpha + (\dot{\partial}^\beta_h v^i)\, \ddot{x}^h_{\beta\alpha}\}\, d\tau\,. \tag{5.2}$$

Let $X^i_\varepsilon(x^h, \dot{x}^h_\alpha)$ be a vector field defined over a region R of the space X_n, whose value at $\bar{x}^i$ is given by $\overline{X}^i_\varepsilon(\bar{x}^h, \dot{\bar{x}}^h_\alpha)$; then the variation in X^i_ε arising from (5.1) and (5.2) will be

$$\overline{X}^i_\varepsilon(\bar{x}^h, \dot{\bar{x}}^h_\theta) - X^i_\varepsilon(x^h, \dot{x}^h_\theta) = \{v^k \partial_k X^i_\varepsilon + (\dot{\partial}^\alpha_h X^i_\varepsilon)(\dot{x}^k_\alpha \partial_k v^h + \ddot{x}^k_{\beta\alpha} \dot{\partial}^\beta_k v^h)\}\, d\tau\,. \tag{5.3}$$

If (5.1) is an infinitesimal coordinate transformation then the variation in X^i_ε arising from such a transformation which takes X^i_ε to $'X^i_\varepsilon(\bar{x}^h, \dot{\bar{x}}^h_\theta)$ will be

$$'X^i_\varepsilon(\bar{x}^h, \dot{\bar{x}}^h_\theta) - X^i_\varepsilon(x^h, \dot{x}^h_\theta) = X^j_\varepsilon(\partial_j v^i)\, d\tau\,. \tag{5.4}$$

The Lie-derivative of a vector field X^ε_i, represented by $\mathcal{L}X^i_\varepsilon$, can be expressed as

$$\mathcal{L}X^i_\varepsilon(x^h, \dot{x}^h_\theta) = \{\overline{X}^i_\varepsilon(\bar{x}^h, \dot{\bar{x}}^h_\theta) - {}'X^i_\varepsilon(\bar{x}^h, \dot{\bar{x}}^h_\theta)\} \mid d\tau\,,$$

which on simplification leads to

$$\mathcal{L}X^i_\varepsilon(x^h, \dot{x}^h_\theta) = v^k \partial_k X^i_\varepsilon - X^k_\varepsilon \partial_k v^i + (\dot{\partial}^\alpha_h X^i_\varepsilon)(\dot{x}^k_\alpha \partial_k v^h + \ddot{x}^k_{\beta\alpha} \dot{\partial}^\beta_k v^h)\,. \tag{5.5}$$

Using equation (1.10) in (5.5) we obtain

$$\begin{aligned}\mathcal{L}X^i_\varepsilon(x^h, \dot{x}^h_\theta) &= v^k X^i_{\varepsilon,k} - X^k_\varepsilon(v^i_{,k} + M^{*h}_{rk} \dot{x}^r_\gamma \dot{\partial}^\gamma_h v^i + 2\, v^l M^{*i}_{[kl]}) + \\ &+ (\dot{\partial}^\alpha_h X^i_\varepsilon)\{v^h_{,k} \dot{x}^k_\alpha + (\dot{\partial}^\beta_l v^h)(\ddot{x}^l_{\beta\alpha} + M^{*l}_{rk} \dot{x}^r_\beta \dot{x}^k_\alpha) + 2\, v^k M^{*h}_{[rk]} \dot{x}^r_\alpha\}\,.\end{aligned} \tag{5.6}$$

Remark. One can observe that if we use $X^l_\varepsilon M^{*i}_{hl}$ in place of the last term of (1.10), then the last term of equation (5.6) vanishes.

Formula (5.6) can be extended to an arbitrary tensor in the usual way. To find the Lie-derivative of the coefficients of connection M^{*i}_{jk} we use the original definition.

For M^{*i}_{jk} we have

$$\overline{M}^{*i}_{jk}(\bar{x}^h, \dot{\bar{x}}^h_\theta) - M^{*i}_{jk}(x^h, \dot{x}^h_\theta) = \{v^r \partial_r M^{*i}_{jk} + (\dot{\partial}^\alpha_h M^{*i}_{jk})(\dot{x}^r_\alpha \partial_r v^h + \ddot{x}^r_{\beta\alpha} \dot{\partial}^\beta_r v^h)\} d\tau \tag{5.7}$$

and

$$'M^{*i}_{jk}(\bar{x}^h, \dot{\bar{x}}^h_\theta) - M^{*i}_{jk}(x^h, \dot{x}^h_\theta) = - d\tau \{M^{*i}_{jr} \partial_k v^r + M^{*i}_{rk} \partial_j v^r + \partial_j \partial_k v^i - M^{*r}_{jk} \partial_r v^i\}. \tag{5.8}$$

From (5.7) and (5.8) we define the Lie-derivative of the coefficients of connection M^{*i}_{jk} as follows:

$$\mathcal{L} M^{*i}_{jk} \stackrel{\text{def}}{=} \{v^r \partial_r M^{*i}_{jk} + M^{*i}_{jr} \partial_k v^r + M^{*i}_{rk} \partial_j v^r + \partial_j \partial_k v^i - M^{*r}_{jk} \partial_r v^i + (\dot{\partial}^\alpha_h M^{*i}_{jk})(\dot{x}^r_\alpha \partial_r v^h + \ddot{x}^r_{\beta\alpha} \dot{\partial}^\beta_r v^h)\}. \tag{5.9}$$

Differentiating (5.1) with respect to u^A we find

$$\overline{B}^i_A = B^i_A + (\partial_A v^i) d\tau, \tag{5.10}$$

which gives

$$\overline{B}^i_A(\bar{x}^h, \dot{\bar{x}}^h_\theta) - B^i_A(x^h, \dot{x}^h_\theta) = B^k_A(\partial_k v^i) d\tau \tag{5.11}$$

and

$$'B^i_A(\bar{x}^h, \dot{\bar{x}}^h_\theta) - B^i_A(x^h, \dot{x}^h_\theta) = B^k_A(\partial_k v^i) d\tau. \tag{5.12}$$

Thus, defining a further operator $\tilde{\mathcal{L}} d\tau = \overline{B}^i_A - {}'B^i_A$, we observe that for the entities of X_n, $\mathcal{L} = \tilde{\mathcal{L}}$ and

$$\tilde{\mathcal{L}} B^i_A = 0. \tag{5.13}$$

Now equation (1.2) together with (5.13) reduces to

$$\tilde{\mathcal{L}} g^{\alpha\gamma}_{AC} = (\mathcal{L} g^{\alpha\gamma}_{ij}) B^i_A B^j_C,$$

wich by virtue of $g^{\alpha\beta}_{AB} g^{BC}_{\beta\gamma} = \delta^C_A \delta^\alpha_\gamma$ gives

$$\tilde{\mathcal{L}} g^{AB}_{\alpha\beta} = - g^{AE}_{\alpha\gamma} g^{FB}_{\delta\beta} \tilde{\mathcal{L}} g^{\gamma\delta}_{EF}; \tag{5.14}$$

therefore we have

$$\tilde{\mathcal{L}} B^A_i = g^{AE}_{\alpha\beta} B^j_E N^h_i (\mathcal{L} g^{\alpha\beta}_{hj}) \tag{5.15}$$

and

$$\tilde{\mathcal{L}} N^j_i = - g^{AE}_{\alpha\beta} B^k_E B^j_A N^h_i (\mathcal{L} g^{\alpha\beta}_{hk}). \tag{5.16}$$

Since we can find

$$'B_j^A(\bar{x}^h, \dot{\bar{x}}_\theta^h) - B_j^A(x^h, \dot{x}_\theta^h) = -B_k^A(\partial_j v^k)\, d\tau, \tag{5.17}$$

therefore from (1.7) and (5.17) we get

$$\bar{B}_j^A(\bar{x}^h, \dot{\bar{x}}_\theta^h) - B_j^A(x^h, \dot{x}_\theta^h) = \{\tilde{\mathscr{L}} N_j^i + (N_j^k \partial_k v^i - N_k^i \partial_j v^k)\}\, d\tau. \tag{5.18}$$

Since we know that $\overset{\circ}{H}{}^i_{AC}$ is normal to X_m, we can observe that $H^i_{\alpha C}$ is also normal to X_m; hence from equations (1.5) and (3.2) we obtain

$$H^i_{\alpha A} = N_k^i (B^k_{CA} + M^{*k}_{jh} B^j_C B^h_A)\, m^C_\alpha. \tag{5.19}$$

Now from equations (5.11), (5.18) and (5.19) we obtain on simplification

$$\bar{H}^i_{\alpha A}(\bar{x}^h, \dot{\bar{x}}_\theta^h) - H^i_{\alpha A}(x^h, \dot{x}_\theta^h) = \{\tilde{\mathscr{L}} N_k^i H^k_{\alpha A} + \partial_k v^i H^k_{\alpha A} + \\ + N_k^i B^h_A B^j_C m^C_\alpha \mathscr{L} M^{*k}_{jh}\}\, d\tau. \tag{5.20}$$

Hence we have

$$\tilde{\mathscr{L}} H^i_{\alpha A} = \tilde{\mathscr{L}} N_k^i H^k_{\alpha A} + N_k^i (\mathscr{L} M^{*k}_{jh}) B^j_C B^h_A m^C_\alpha, \tag{5.21}$$

which by virtue of (5.13) gives

$$\tilde{\mathscr{L}} M^h_{jA} = (\mathscr{L} M^{*h}_{ji}) B^i_A + (\mathscr{L} A^{h\alpha}_{ji}) H^i_{\alpha A} + H^{h\alpha}_{ji} \tilde{\mathscr{L}} H^i_{\alpha A}. \tag{5.22}$$

From equation (2.7) we can find

$$\tilde{\mathscr{L}} \overline{M}^{*A}_{DC} = \tilde{\mathscr{L}} B^A_i (B^i_{DC} + M^i_{hC} B^h_D) + (\tilde{\mathscr{L}} M^j_{hC}) B^h_D B^A_j. \tag{5.23}$$

Using (3.10) and (5.15) in (5.23) we get

$$\tilde{\mathscr{L}} \overline{M}^{*A}_{DC} = \overset{\circ}{H}{}^i_{CD} \tilde{\mathscr{L}} B^A_i + (\tilde{\mathscr{L}} M^i_{hC}) B^h_D B^A_j, \tag{5.24}$$

which by virtue of (5.13) and (5.16) leads to

$$\tilde{\mathscr{L}} \overset{\circ}{H}{}^i_{BC} = \tilde{\mathscr{L}} N^i_j \overset{\circ}{H}{}^j_{BC} + N^i_j (\tilde{\mathscr{L}} M^j_{hB}) B^h_C. \tag{5.25}$$

Let X^i_ε and X^A_ε be the corresponding vectors in X_n and X_m and let the vectors obtained by parallel displacements be given by $'X^i_\varepsilon$ and $'X^A_\varepsilon$; then for $V^i_\varepsilon \overset{\text{def}}{=} {'X^i_\varepsilon} - B^i_A\, {'X^A_\varepsilon}$, by virtue of equations (3.9), (3.10) and (3.23) we find on simplification

$$V^i_\varepsilon = -\overset{|}{H}{}^{i\alpha}_{BC} X^B_\varepsilon \overline{D} m^C_\alpha - \overset{\circ}{H}{}^i_{BC} X^B_\varepsilon\, du^C, \tag{5.26}$$

which gives the geometrical interpretation of the two quadratic forms.

Substituting the values on the right hand side of (5.14) and simplifying we can write

$$\tilde{\mathscr{L}} g^{\alpha\gamma}_{AC} = 2\{B^j_{(A} \overset{\circ}{D}_{C)} v^h - H^r_{\beta(C} B^j_{A)} A^{h\beta}_{kr} v^k + B^{ij}_{AC} A^{h\beta}_{ir} v^r_{,k} m^k_\beta\}\, g^{\alpha\gamma}_{jh}. \tag{5.27}$$

If the displacement (5.1) is such that v^i is normal to X_m then we can easily obtain

$$g^{\alpha\beta}_{jh}\, B^j_A\, \mathring{D}_C\, v^h = -\, g^{\alpha\beta}_{jh}\, \mathring{H}^j_{CA}\, v^h, \tag{5.28}$$

which by virtue of (5.27) leads to

$$\tilde{\mathscr{L}} g^{\alpha\gamma}_{AC} = -\, 2\, v^j \{\mathring{H}^j_{(AC)} + A^{j\beta}_{ik}\, B^k_{(A}\, \mathring{H}^i_{\beta C)}\} + 2\, A^{\alpha\gamma\delta}_{ijh}\, v^h_{,k}\, m^k_\delta\, B^{ij}_{AC}, \tag{5.29}$$

which is the second geometrical interpretation of the coefficients $\mathring{H}^i_{AC}$ and is a generalisation of the result of Davies [1].

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