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Decompositions of recurrent conformal and Weyl's projective curvature tensors

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Geometria differenziale. — *Decompositions of recurrent conformal and Weyl's projective curvature tensors.* Nota di SHRI KRISHNA DEO DUBEY, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — In analogia con quanto già effettuato da Takano [4], Sinha e Singh [3] Singh [2], qui si ottengono varie decomposizioni dei tensori ricorrenti di curvature $R^{*\cdots i}_{jkl}$, $C^{*\cdots i}_{jkl}$ e W^{i}_{jkl} in uno spazio speciale di Kawaguchi.

1. INTRODUCTION

In an n-dimensional special Kawaguchi space K_n of order 2, the arc length of a curve $x^i = x^i(t)$ is given by the integral (Kawaguchi [1])

$$s = \int [A_i(x, x') x''^i + B(x, x')]^{1/p} \, dt \quad , \quad p \neq 0 \quad , \quad 3/2, \tag{1.1}$$

where $x'^i = dx^i/dt$ and $x''^i = d^2 x^i/dt^2$.

Let v^i be a contravariant vector field homogeneous of degree zero with respect to x'^i. The covariant derivatives of v^i are defined by ([1])

$$\nabla_j v^i = \partial_j v^i - v^i_{(k)} \Gamma^k_{(j)} + \Gamma^i_{(k)(j)} v^k,$$

$$\nabla'_j v^i = v^i_{(j)} = \frac{\partial v^i}{\partial x'^j}, \qquad (\partial_j = \partial/\partial x^j).$$

The conformal curvature tensor $C^{*\cdots i}_{jkl}$ in a special Kawaguchi space is defined as

$$C^{*\cdots i}_{jkl} = R^{*\cdots i}_{jkl} - \tag{1.2}$$

$$- \frac{\delta^i_l}{n+1} S^*_{jk} + \frac{\delta^i_k}{n-1}\left(R^*_{jl} - \frac{1}{n+1} S^*_{lj}\right) - \frac{\delta^i_j}{n-1}\left(R^*_{kl} - \frac{1}{n+1} S^*_{lk}\right),$$

where

$$R^{*\cdots i}_{jkl} = \frac{\partial \Pi^i_{lj}}{\partial x^k} - \frac{\partial \Pi^i_{lk}}{\partial x^j} + \Pi^h_{lj} \Pi^i_{kh} - \Pi^h_{lk} \Pi^i_{jh} + \Pi^h_j \Pi^i_{lk(h)} - \Pi^h_{(k)} \Pi^i_{lj(h)}, \tag{1.3}$$

$$R^*_{kl} = R^{*\cdots a}_{akl} \quad , \quad S^*_{jk} = R^{*\cdots a}_{jka}, \tag{1.4}$$

and

$$R^{*\cdots i}_{jkl} = - R^{*\cdots i}_{kjl}. \tag{1.5}$$

(*) Nella seduta del 23 giugno 1977.

Also, we have

(1.6) $$C^{*\cdots i}_{jkl} + C^{*\cdots i}_{klj} + C^{*\cdots i}_{ljk} = 0,$$

(1.7) $$C^{*\cdots i}_{jkl} + C^{*\cdots i}_{kjl} = 0$$

and

(1.8) $$C^{*\cdots i}_{jk} = C^{*\cdots i}_{jkl}\, x'^{l}.$$

The Weyl tensor in a special Kawaguchi space is expressed as

(1.9) $$W^i_k = H^i_k - H\delta^i_k - \frac{1}{n+1}\left(\frac{\partial H^a_k}{\partial x'^a} - \frac{\partial H}{\partial x'^k}\right) x'^i,$$

where

(1.10) $$H^i_k = K^{\cdots i}_{jk}\, x'^j \quad , \quad H = \frac{1}{n-1} H^i_i .$$

The Weyl projective curvature tensors have the following properties:

(1.11) $$W^i_{jkl} + W^i_{klj} + W^i_{ljk} = 0,$$

(1.12) $$W^i_{jk}\, x'^j = W^i_k \quad , \quad W^i_{jkl}\, x'^l = W^i_{jk},$$

(1.13) $$W^i_{jkl} = - W^i_{kjl},$$

(1.14) $$W^i_k = C^{*\cdots i}_{jkl}\, x'^j\, x'^l,$$

(1.15) $$W^i_{jkl} = W^i_{jk(l)},$$

(1.16) $$W^i_i = 0 \quad , \quad W^i_k\, x'^k = 0 \quad , \quad W^i_{k(i)} = 0 .$$

The curvature tensor $R^{*\cdots i}_{jkl}$ in a special Kawaguchi space is said to be recurrent or bi-recurrent, if it satisfies the conditions

(1.17) $$\nabla_m R^{*\cdots i}_{jkl} = v_m R^{*\cdots i}_{jkl} \quad , \quad v_m \neq 0$$

or

(1.18) $$\nabla_p \nabla_m R^{*\cdots i}_{jkl} = \alpha_{pm} R^{*\cdots i}_{jkl} \qquad (R^{*\cdots i}_{jkl} \neq 0)$$

respectively, in which v_m and α_{pm} are the recurrence vector field and the recurrence tensor field.

Equations (1.2), (1.4), (1.17), (1.18) yield that the curvature tensor $C^{*\cdots i}_{jkl}$ is recurrent and bi-recurrent with the same recurrence vector field and recurrence tensor field as in the case of $R^{*\cdots i}_{jkl}$, that is,

(1.19) $$\nabla_m C^{*\cdots i}_{jkl} = v_m C^{*\cdots i}_{jkl} \quad , \quad (C^{*\cdots i}_{jkl} \neq 0)$$

and

(1.20) $$\nabla_p \nabla_m C^{*\cdots i}_{jkl} = \alpha_{pm} C^{*\cdots i}_{jkl} \quad , \quad (C^{*\cdots i}_{jkl} \neq 0) .$$

The Weyl projective curvature tensor W^i_{jkl} in a special Kawaguchi space is said to be recurrent or bi-recurrent, if it satisfies the conditions

$$\nabla_m W^i_{jkl} = \lambda_m W^i_{jkl} \qquad (W^i_{jkl} \neq 0) \tag{1.21}$$

or

$$\nabla_q \nabla_m W^i_{jkl} = a_{qm} W^i_{jkl} \qquad (W^i_{jkl} \neq 0) \tag{1.22}$$

respectively, where λ_m and a_{qm} are the recurrence vector field and the recurrence tensor field.

2. Decomposition of recurrent curvature tensor $R^{*\cdots i}_{jkl}$

We assume that the decomposition of the recurrent curvature tensor $R^{*\cdots i}_{jkl}$ has the following form

$$R^{*\cdots i}_{jkl} = r^i \varepsilon_{jkl} , \tag{2.1}$$

where ε_{jkl} is a non zero decomposed tensor field and r^i is a non zero vector field satisfying the condition

$$r^m v_m = 1 , \tag{2.2}$$

in which v_m is the recurrence vector field.

We suppose that the curvature tensor is recurrent of the first order.

Equations (1.17) and (2.1) yield

$$(\nabla_m r^i) \varepsilon_{jkl} + r^i \nabla_m \varepsilon_{jkl} = v_m r^i \varepsilon_{jkl} . \tag{2.3}$$

If we suppose that $(\nabla_m r^i) = 0$, then (2.3) can be written as

$$r^i (\nabla_m \varepsilon_{jkl} - v_m \varepsilon_{jkl}) = 0 . \tag{2.4}$$

Since $r^i \neq 0$

$$\nabla_m \varepsilon_{jkl} = v_m \varepsilon_{jkl} , \tag{2.5}$$

which gives the following:

THEOREM (2.1). *If the recurrent curvature tensor $R^{*\cdots i}_{jkl}$ has the decomposition (2.1) and the vector field r^i satisfies the condition $\nabla_m r^i = 0$ then the decomposed tensor field ε_{jkl} is recurrent with the same recurrence vector field as the tensor $R^{*\cdots i}_{jkl}$.*

THEOREM (2.2). *If $r^i = \dot{x}^i$ and the recurrent curvature tensor $R^{*\cdots i}_{jkl}$ has the decomposition (2.1) then the decomposed tensor field ε_{jkl} is recurrent with the same recurrence vector field as the tensor $R^{*\cdots i}_{jkl}$.*

Equations (1.5) and (2.1) yield

$$\varepsilon_{jkl} = -\varepsilon_{kjl} \,. \tag{2.6}$$

Using the fact that $\Pi^i_{jk} = \Pi^i_{kj}$ and equation (1.3), we have

$$R^{*\cdots i}_{jkl} + R^{*\cdots i}_{klj} + R^{*\cdots i}_{ljk} = 0 \,. \tag{2.7}$$

Equations (2.1) and (2.7) yield

$$\varepsilon_{jkl} + \varepsilon_{klj} + \varepsilon_{ljk} = 0 \,. \tag{2.8}$$

Contracting the indices i, l and i, j in equation (2.1) and using (1.4), we get

$$S^*_{jk} = r^a \, \varepsilon_{jka} \tag{2.9}$$

and

$$R^*_{kl} = r^a \, \varepsilon_{akl} \,. \tag{2.10}$$

THEOREM (2.3). *If the recurrent curvature tensor* $R^{*\cdots i}_{jkl}$ *is decomposed with the tensor field* ε_{jkl} *then a sufficient condition in order that* $R^{*\cdots i}_{jkl}$ *is equal to the conformal curvature tensor* $C^{*\cdots i}_{jkl}$ *is that the relation*

$$\delta^i_l \,(n-1)\, \varepsilon_{jka} + (n+1)\,(\delta^i_j \, \varepsilon_{akl} - \delta^i_k \, \varepsilon_{ajl}) + \delta^i_k \, \varepsilon_{lja} - \delta^i_j \, \varepsilon_{kla} = 0 \tag{2.11}$$

holds.

Proof. Equations (1.2), (2.1), (2.9) and (2.10) yield

$$C^{*\cdots i}_{jkl} = r^i \, \varepsilon_{jkl} - \tag{2.12}$$

$$- \frac{r^a}{(n+1)\,(n-1)} \left[\delta^i_l \,(n-1)\, \varepsilon_{jka} - \delta^i_k \,(n+1)\, \varepsilon_{ajl} + \right.$$

$$\left. + \delta^i_k \, \varepsilon_{lja} + \delta^i_j \,(n+1)\, \varepsilon_{akl} - \delta^i_j \, \varepsilon_{kla}\right] .$$

The proof of the above theorem is an immediate consequence of equations (2.1), (2.11) and (2.12).

THEOREM (2.4). *If the bi-recurrent curvature tensor* $R^{*\cdots i}_{jkl}$ *has the decomposition* (2.1) *and the vector field* r^i *satisfies the condition* $\nabla_p \nabla_m r^i = 0$ *then the decomposed tensor field* ε_{jkl} *is bi-recurrent with the same bi-recurrence tensor field as the tensor* $R^{*\cdots i}_{jkl}$.

Proof. Equations (1.18) and (2.1) yield

$$r^i \,(\nabla_p \nabla_m \, \varepsilon_{jkl} - \alpha_{pm} \, \varepsilon_{jkl}) + \varepsilon_{jkl} \,(\nabla_p \nabla_m \, r^i) = 0 \,. \tag{2.13}$$

Using the relation $\nabla_p \nabla_m r^i = 0$ and the fact that $r^i \neq 0$, we find that ε_{jkl} is bi-recurrent with the bi-recurrence tensor field α_{pm}.

3. Decompositions of recurrent conformal curvature tensors

We suppose that the decomposition of the recurrent conformal curvature tensor $C^{*\cdots i}_{jkl}$ has the following form:

$$C^{*\cdots i}_{jkl} = s^i \rho_{jkl} , \tag{3.1}$$

where ρ_{jkl} is a non zero decomposed tensor field and s^i is a non zero vector field satisfying the condition

$$s^i v_m = 1 , \tag{3.2}$$

in which v_m is the recurrence vector field.

Equations (1.6), (1.7) and (3.1) yield

$$\rho_{jkl} + \rho_{klj} + \rho_{ljk} = 0 , \tag{3.3}$$

$$\rho_{jkl} + \rho_{kjl} = 0 . \tag{3.4}$$

Multiplying equation (3.1) by $\acute{x}^l$ and using (1.8), we get

$$C^{*\cdots i}_{jk} = s^i \rho_{jk} , \tag{3.5}$$

where

$$\rho_{jk} = \rho_{jkl} \acute{x}^l . \tag{3.6}$$

Equations (1.19) and (3.1) yield

$$(\nabla_m \rho_{jkl} - v_m \rho_{jkl}) s^i + (\nabla_m s^i) \rho_{jkl} = 0 . \tag{3.7}$$

The following theorems are an immediate consequence of equation (3.7):

THEOREM (3.1). *If the recurrent conformal curvature tensor* $C^{*\cdots i}_{jkl}$ *has the decomposition* (3.1) *and the vector field* s^i *satisfies the condition* $\nabla_m s^i = 0$ *then the decomposed tensor field* ρ_{jkl} *is recurrent with the same recurrence vector field as the tensor* $C^{*\cdots i}_{jkl}$.

THEOREM (3.2). *If the recurrent conformal curvature tensor* $C^{*\cdots i}_{jkl}$ *has the decomposition* $C^{*\cdots i}_{jkl} = \acute{x}^i \rho_{jkl}$, *then the decomposed tensor* ρ_{jkl} *is recurrent with the same recurrence vector field as the tensor* $C^{*\cdots i}_{jkl}$.

Equations (1.20) and (3.1) give

$$s^i (\nabla_p \nabla_m \rho_{jkl} - \alpha_{pm} \rho_{jkl}) + \rho_{jkl} \nabla_p \nabla_m s^i = 0 . \tag{3.8}$$

Using the relation $\nabla_p \nabla_m s^i = 0$ and the fact $s^i \neq 0$, we find that

$$\nabla_p \nabla_m \rho_{jkl} = \alpha_{pm} \rho_{jkl} . \tag{3.9}$$

Thus, we have

THEOREM (3.3). *If the bi-recurrent conformal curvature tensor has the decomposition* (3.1) *and the vector field* s^i *satisfies the condition* $\nabla_p \nabla_m s^i = 0$ *then the decomposed tensor field* ρ_{jkl} *is bi-recurrent with the same bi-reccurrence tensor field as the tensor* $C^{*\cdots i}_{jkl}$.

We suppose that the recurrent conformal curvature tensor $C^{*\cdots i}_{jkl}$ has the decomposition in the following form:

$$C^{*\cdots i}_{jkl} = X^i_j \psi_{kl} , \tag{3.10}$$

where $\psi_{kl}(x, \dot{x})$ is a decomposed tensor field and $X^i_j(x, \dot{x})$ is a tensor field.

THEOREM (3.4). *If the recurrent conformal curvature tensor has the decomposition* (3.10), *then the following identity holds*:

$$p_j \psi_{kl} + p_k \psi_{lj} + p_l \psi_{jk} = 0 , \tag{3.11}$$

where

$$p_j = X^i_j v_i . \tag{3.12}$$

Proof. Equations (1.6) and (3.10) yield

$$X^i_j \psi_{kl} + X^i_k \psi_{lj} + X^i_l \psi_{jk} = 0 . \tag{3.13}$$

Multiplying (3.13) by the recurrence vector field v_i and using (3.12), we obtain the identity (3.11).

Multiplying equation (3.10) by the recurrence vector field v_i and using relation (3.12) we get

$$v_i C^{*\cdots i}_{jkl} = p_j \psi_{kl} . \tag{3.14}$$

Equations (1.7) and (3.10) give

$$X^i_j \psi_{kl} = - X^i_k \psi_{jl} . \tag{3.15}$$

Multiplying (3.15) by v_i and using (3.12), we get

$$p_j \psi_{kl} = - p_k \psi_{jl} . \tag{3.16}$$

Equations (3.11) and (3.16) yield the following:

THEOREM (3.5). *If the recurrent conformal curvature tensor has the decomposition* (3.10), *then the identity*

$$p_k \psi_{lj} = p_j (\psi_{lk} - \psi_{kl}) \tag{3.17}$$

holds.

Equations (1.19) and (3.10) give

$$X^i_j (\nabla_m \psi_{kl} - v_m \psi_{kl}) + (\nabla_m X^i_j) \psi_{kl} = 0 . \tag{3.18}$$

We assume that $\nabla_m X^i_j = 0$. Since $X^i_j \neq 0$, equation (3.18) gives the following:

THEOREM (3.6). *If the recurrent conformal curvature tensor has the decomposition* (3.10) *and the tensor field* X^i_j *satisfies the condition* $\nabla_m X^i_j = 0$ *then the decomposed tensor field* ψ_{kl} *is recurrent with the same recurrence vector field as the tensor* $C^{*\cdots i}_{jkl}$.

In a similar way, equations (1.20) and (3.10) yield the following:

THEOREM (3.7). *If the bi-recurrent conformal curvature tensor has the decomposition* (3.10) *and the tensor field* X^i_j *satisfies the condition* $\nabla_p \nabla_m X^i_j = 0$ *then the decomposed tensor field* ψ_{kl} *is recurrent with the same bi-recurrence tensor field as the tensor* $C^{*\cdots i}_{jkl}$.

4. Decomposition of recurrent Weyl's projective curvature tensors

We suppose that the Weyl projective curvature tensor has the following decomposition:

$$W^i_{jkl} = \xi^i \sigma_{jkl} , \tag{4.1}$$

where σ_{jkl} is a non zero decomposed tensor field and ξ^i is a non zero vector field satisfying the condition

$$\xi^i \lambda_i = 1 , \tag{4.2}$$

λ_i being the recurrence vector field.

Equations (1.11), (1.13) and (4.1) give

$$\sigma_{jkl} + \sigma_{klj} + \sigma_{ljk} = 0 , \tag{4.3}$$

$$\sigma_{jkl} = - \sigma_{kjl} . \tag{4.4}$$

Multiplying equation (4.1) by $\overset{\prime}{x}{}^l$ then by $\overset{\prime}{x}{}^j$ and using (1.12), we get

$$W^i_{jk} = \xi^i \sigma_{jk} , \tag{4.5}$$

$$W^i_k = \xi^i \sigma_k , \tag{4.6}$$

in which we have used the notations:

$$\sigma_{jk} = \sigma_{jkl} \overset{\prime}{x}{}^l , \tag{4.7}$$

$$\sigma_k = \sigma_{jk} \overset{\prime}{x}{}^j . \tag{4.8}$$

We consider the conformal and Weyl's curvature tensors having the decomposition (3.1) and (4.1) respectively. Putting

$$\rho_k = \rho_{jkl}\, \acute{x}^j \acute{x}^l , \tag{4.9}$$

equations (1.14), (3.1), (4.1) and (4.6) yield the relation

$$\xi^i \sigma_k = \rho_k s^i . \tag{4.10}$$

Thus, we have

THEOREM (4.1). *If the conformal and Weyl's curvature tensor have the decomposition* (3.1) *and* (4.1) *respectively and* $\xi^i = s^i$ *then the vector fields* ρ_k *and* σ_k *are equal.*

THEOREM (4.2). *If the recurrent Weyl's projective curvature tensor has the decomposition* (4.1) *and the vector field* ξ^i *satisfies the condition* $\nabla_m \xi^i = 0$ *then the decomposed tensor field* σ_{jkl} *is recurrent with the same recurrence vector field as the tensor* W^i_{jkl}.

THEOREM (4.3). *If the recurrent Weyl's projective curvature tensor has the decomposition*

$$W^i_{jkl} = \acute{x}^i \sigma_{jkl} ,$$

then the decomposed tensor field σ_{jkl} *is recurrent with the same recurrence vector field as the tensor* W^i_{jkl}.

Differentiating equation (4.5) covariantly with respect to $\acute{x}^l$ and using the equations (1.15) and (4.1), we get

$$\xi^i \sigma_{jkl} = \xi^i_{(l)} \sigma_{jk} + \xi^i \sigma_{jk(l)} . \tag{4.11}$$

Equations (1.16), (1.12), (4.7) and (4.11) give the following:

THEOREM (4.4). *If the recurrent Weyl's projective curvature tensor* W^i_{jk} *has the decomposition* (4.1) *and the vector field* ξ^i *is positively homogeneous of degree zero in* $\acute{x}^l$ *then* σ_{jk} *is positively homogeneous of the first degree in* $\acute{x}^i$.

We suppose that the Weyl's projective curvature tensor W^i_{jkl} is recurrent and bi-recurrent with the recurrent vector field λ_m and bi-recurrence tensor field a_{pm} and $\xi^i = \acute{x}^i$. Differentiating equation (4.1) covariantly with respect to x^m, using (1.21), we get

$$\nabla_m \sigma_{jkl} = \lambda_m \sigma_{jkl} . \tag{4.12}$$

Again, differentiating equation (4.12) covariantly with respect to x^p and using (1.22), we get

$$\nabla_p \nabla_m \sigma_{jkl} = a_{pm} \sigma_{jkl} . \tag{4.13}$$

Thus, we have

THEOREM (4.5). *If the recurrent, bi-recurrent Weyl's projective curvature tensor has the decomposition*

$$W^i_{jkl} = \sigma_{jkl} x'^l$$

then the decomposed tensor field σ_{jkl} *is recurrent and bi-recurrent with the recurrence vector field* λ_m *and bi-recurrence tensor field* a_{pm} *which are also the recurrence vector field and bi-recurrence tensor field of* W^i_{jkl}.

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