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Equazioni funzionali. — *Asymptotic Behavior of Solutions of Nonlinear Functional Equations via Nonstandard Analysis.* Nota di

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RIASSUNTO. — Gli Autori usano speciali tecniche per trovare alcune proprietà caratteristiche delle soluzioni delle equazioni

$$L_n x(t) + \delta f(t, x[g_1(t)], \cdots, x[g_m(t)]) = h(t) \quad , \quad \delta = \pm 1.$$

1. INTRODUCTION

Nonstandard analysis was introduced in oscillatory theory by Komkov and Waid [1] and Komkov [2]. Recently, the Authors [3] improved their results and gave some new criteria for the asymptotic behavior of solutions of the following n-th order differential equation with deviating arguments

$$x^{(n)}(t) + \delta a(t) G(x[g_1(t)], \cdots, x[g_m(t)]) = h(t) \quad , \quad \delta = \pm 1.$$

In this Note, we extend these results to the more general differential equation

$$E(\delta) \qquad L_n x(t) + \delta f(t, x[g_1(t)], \cdots, x[g_m(t)]) = h(t) \quad , \quad \delta = \pm 1$$

by using nonstandard techniques, in the frame-work of Robinson's theory [4, 5]. Here L_n is an operator defined by

$$L_0 x(t) = x(t) \quad , \quad L_i x(t) = \frac{1}{r_i(t)} \frac{d}{dt} L_{i-1} x(t) \quad , \quad r_n(t) = 1,$$

for $i = 1, \cdots, n$.

Let R^* denote the nonstandard extension of the real line R, which has the property that sentences formulated in language L are true in R^* if and only if they are true in R (see [5]). We see that R is a subset of R^* and R^* also contains infinitesimal numbers and infinite numbers which are not in R. An infinite positive (resp. negative) number is a nonstandard number which is greater (resp. smaller) than any real number. We shall denote by $R^*_{+\infty}$ and $R^*_{-\infty}$, respectively, the set of the infinite positive and negative numbers. The

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reciprocal of an infinite number is called an infinitesimal number. If x is a real number, then we call x a standard number of R^*, otherwise it is called a nonstandard number. Let R^*_{bd} denote the set of the elements of R^* which are bounded in absolute value by a standard number. If x, y are elements of R^* such that $x - y$ is an infinitesimal, we shall say that x is infinitely close to y, and denote this by $x =_1 y$.

For related results, we refer to Saito [6], Stroyan and Luxemburg [7].

Let $I \equiv [t_0, \infty)$ for some fixed $t_0 > 0$. Throughout this paper, we assume that the following two conditions always hold:

$$(a) \quad r_i, g_j, h \in C[I, R], r_i(t) > 0, \int_{t_0}^{\infty} r_i(t)\, dt = \infty \text{ and } \lim_{t\to\infty} g_j(t) = \infty$$

for $i = 1, \cdots, n$ and $j = 1, \cdots, m$.

(b) $f \in C[I \times R^m, R]$.

We need the following four lemmas. The first is due to Robinson [4], and the others are due to Komkov and Waid [1].

LEMMA 1. *$\int_{t_0}^{\infty} g(t)\, dt$ converges if and only if $\int_{t_1}^{t_2} g(t)\, dt =_1 0$ for any $t_1, t_2 \in R^*_{+\infty}$* ([4], p. 75).

LEMMA 2. *A standard function $x(t)$, $t \in I$, is oscillatory if and only if $x(t)$, $t \in R^*$, vanishes for some $t \in R^*_{+\infty}$.*

LEMMA 3. *A standard function $x(t)$ is unbounded if and only if $|x(t)| \in R^*_{+\infty}$ for some $t \in R^*_{+\infty}$.*

LEMMA 4. *Let $\lim_{t\to\infty} \int_{t_0}^{t} g(s)\, ds = +\infty\ (-\infty)$. Then for any $A \in R^*, A > 0$ (resp. < 0), and any $t_1 > t_0$, $t_1 \in R^*$, there exists $t_2 \in R^*, t_2 > t_1$, such that $\int_{t_1}^{t_2} g(t)\, dt > A$ (resp. $< A$). Moreover, for any $t_3 \in R^*_{bd}$, $t_4 \in R^*_{+\infty}$ (resp. $R^*_{-\infty}$), we have $\int_{t_3}^{t_4} g(t)\, dt \in R^*_{+\infty}$ (resp. $R^*_{-\infty}$).*

2. MAIN RESULTS

THEOREM 1. *Let*

(C_1) *$f(t, y_1, \cdots, y_m)$ be a nondecreasing function with respect to $y_1, \cdots, y_m$ and*

$$0 < f(t, y_1, \cdots, y_m) \leq -f(t, -y_1, \cdots, -y_m)$$

for $y_i > 0$, $i = 1, \cdots, m$,

$$(C_2) \qquad \int_{t_0}^{\infty} h(t)\, dt \quad \textit{converge,}$$

and

$$(C_3) \qquad \int_{t_0}^{\infty} |f(t, c, \cdots, c)|\, dt = \infty$$

for any nonzero constant c. *Then every nonoscillatory solution of* E (1) *cannot be bounded away from zero.*

Proof. Assume, to the contrary, that there exists a solution $x(t)$ of E (1) such that $x(t)$ is bounded away from zero on I. Without loss of generality, we assume that $x(t) > c > 0$ for some standard number c. Condition (*a*) implies that there exists a $t_1 > t_0$ such that

$$x[g_i(t)] > c$$

for $t > t_1$ and $i = 1, \cdots, m$. Hence, by (*b*) and (C_1), we have

$$(1) \qquad f(t, x[g_1(t)], \cdots, x[g_m(t)]) \geq f(t, c, \cdots, c)$$

for $t \geq t_1$ and particularly for all $t \in R^*_{+\infty}$. It follows from (C_2) and Lemma 1 that

$$\int_{\xi}^{\eta} h(t)\, dt =_1 0$$

for any $\xi, \eta \in R^*_{+\infty}$. Hence

$$(2) \qquad \int_{\xi}^{\eta} h(t)\, dt < 1.$$

By (1), (2) and the fundamental theorem of calculus

$$(3) \qquad L_{n-1} x(\eta) = L_{n-1} x(\xi) + \int_{\xi}^{\eta} [h(t) - f(t, x[g_1(t)], \cdots, x[g_m(t)])]\, dt$$

$$< L_{n-1} x(\xi) + 1 - \int_{\xi}^{\eta} f(t, c, \cdots, c)\, dt.$$

Regarding ξ as fixed, by (C_3) and Lemma 4, we can choose η so that

$$(4) \qquad \int_{\xi}^{\eta} f(t, c, \cdots, c)\, dt > [2 + L_{n-1} x(\xi)].$$

From (3) and (4), we have

$$L_{n-1}\, x(\eta) < -1 \tag{5}$$

for all η satisfying (4). Since $x(t)$ is positive, (5) and (*a*) imply that $x(t)$ changes sign for some $t \in R^*_{+\infty}$. Therefore, by Lemma 2, $x(t)$ is oscillatory, a contradiction. This contradiction proves our theorem.

COROLLARY. *Under the assumptions of Theorem 1, every solution* $x(t)$ *of* E (1) *is oscillatory or such that* $\liminf_{t\to\infty} |x(t)| = 0$.

Example 1. The equation

$$(t^{-1/2}\, x'(t))' + 4^{-1}\, t^{-2}\, x(t) = 4^{-1}\, t^{-3/2} - 2^{-1}\, t^{-2} \tag{6}$$

satisfies the conditions (C_1) and (C_2), but does not satisfy (C_3). This equation has a nonoscillatory solution $x(t) = t^{1/2}$ which is bounded away from zero.

Example 2. The differential equation

$$(e^{-t}\, x')' + x(t) = e^{-2t} (\sin t - 3 \cos t) + e^{-t} \sin t \tag{7}$$

satisfies all the conditions of Theorem 1. Hence every solution $x(t)$ of (7) is oscillatory or such that $\liminf_{t\to\infty} |x(t)| = 0$. In fact, $x(t) = e^{-t} \sin t$ is an oscillatory solution of (7).

THEOREM 2. *Let* (C_1), $\lim_{t\to\infty} h(t) = 0$ *and the following condition hold*:

$$f(t, y_1, \cdots, y_m) = p(t)\, F(y_1, \cdots, y_m) \tag{C_4}$$

where $p(t) \in C[I, (0, \infty)]$. *If*

$$\liminf_{t\to\infty} p(t) \equiv c > 0\,, \tag{C_5}$$

then every solution $x(t)$ *of* E (1) *is oscillatory or such that* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of E (1). Without loss of generality, we assume that $x(t) > 0$ for all $t \in R^*_{+\infty}$. If $x[g_i(t)] \neq_1 0$ for some $t_1 \in R^*_{+\infty}$, $i = 1, \cdots, m$, then, by (C_1)

$$p(t_1)\, F(x[g_1(t_1)], \cdots, x[g_m(t_1)]) \neq_1 0\,.$$

It follows from E (1) that $L_n\, x(t_1) < 0$ and $L_n\, x(t_1) \neq_1 0$. We see that there must exist $t_2 \in R^*_{+\infty}$, $t_2 > t_1$, such that

$$L_n\, x(t) < 0 \quad \text{and} \quad L_n\, x(t) =_1 0$$

for $t \geq t_2$. Otherwise $L_n x(t)$ is negative and bounded away from zero for $t \geq t_1$. By the condition (a), $x(t)$ must eventually become negative, a contradiction. But $L_n x(t) =_1 0$ for $t \geq t_2$ implies

$$p(t) F(x[g_1(t)], \cdots, x[g_m(t)]) =_1 0,$$

thus, by (C_5),

$$F(x[g_1(t)], \cdots, x[g_m(t)]) =_1$$

which implies $x[g_i(t)] =_1 0$, i.e. $x(t) = {}_1 0$ for $t \in R^*_{+\infty}$.

Example 3. The differential equation

$$(8) \qquad (t(t(tx')')')' + t[x(\log t)]^3 = (t^3 - 6t^2 + 7t - 1)e^{-t} - t^2$$

satisfies every condition of Theorem 2. Hence, every solution $x(t)$ of (8) is oscillatory or tends to zero as $t \to \infty$. In fact, $x(t) = e^{-t}$ is a nonoscillatory solution of (8) which tends to zero as $t \to \infty$.

Example 4. From Example 1, we see that $x(t) = t^{1/2}$ is an unbounded solution of (6). Here $p(t) = 4^{-1} t^{-2}$ does not satisfy the condition (C_5).

THEOREM 3. *Let* (C_4), (C_5) *and*

$$(C_6) \qquad \lim_{t \to \infty} \frac{h(t)}{p(t)} = +\infty$$

hold. Then every solution of $E(\delta)$ *is unbounded.*

Proof. Assume, to the contrary, that there exists a solution $x(t)$ of $E(\delta)$ which is bounded. Then $x[g_i(t)]$ is bounded for $i = 1, \cdots, m$. Since

$$(9) \qquad 2c^{-1} L_n x(t) > \frac{L_n x(t)}{p(t)} = \frac{h(t)}{p(t)} - \delta F(x[g_1(t)], \cdots, x[g_m(t)]),$$

$L_n x(t)$ must be of positive sign for all $t \in R^*_{+\infty}$. If $L_n x(t_1) =_1 0$ for some $t_1 \in R^*_{+\infty}$, then we have

$$(10) \qquad \delta F(x[g_1(t_1)], \cdots, x[g_m(t_1)]) =_1 \frac{h(t_1)}{p(t_1)},$$

which, by (C_6), is an infinite positive number. Since $x[g_i(t)]$ is bounded for $i = 1, \cdots, m$, the left hand side of (10) is bounded, a contradiction. If $L_n x(t) \neq_1 0$ for all $t \in R^*_{+\infty}$, it follows from (9) that $L_n x(t)$ is an infinite positive number for all $t \in R^*_{+\infty}$. This and the condition (a) imply $x(t)$ is an infinite number for all $t \in R^*_{+\infty}$, a contradiction. Thus the proof is complete.

Example 5. The equation

$$(11) \qquad (t^{-1}(t^{-1/2}x')')' + x(t) = t^{1/2} + \frac{3}{2} t^{-4}$$

satisfies the conditions of Theorem 3. Thus, every solution of (11) is unbounded. In fact, this equation has an unbounded solution $x(t) = t^{1/2}$.

THEOREM 4. *Let* (C_1) *and* (C_4) *hold. If*

$$(C_7) \qquad \liminf_{t\to\infty} p(t) \geq c > 0$$

$$(C_8) \qquad \liminf_{t\to\infty} \frac{h(t)}{p(t)} \geq r > 0,$$

then no nonoscillatory solution of $E(\delta)$ *approaches zero as* $t \to \infty$.

Proof. We only prove the case $E(1)$. Let $x(t)$ be a nonoscillatory solution of $E(1)$ which approaches zero. Then there exists a $t_1 \geq t_0$ such that for all $t \geq t_1$

$$F(x[g_1(t)], \cdots, x[g_m(t)]) < 4^{-1} r.$$

Since

$$2c^{-1} L_n x(t) > \frac{L_n x(t)}{p(t)} = -F(x[g_1(t)], \cdots, x[g_m(t)]) + \frac{h(t)}{p(t)}$$
$$> -4^{-1} r + 2^{-1} = 4^{-1} r > 0$$

for $t \geq t_1$, $x(t)$ is an infinite positive number for $t \in R^*_{+\infty}$, a contradiction. This contradiction completes our proof.

Example 6. The equation

$$(12) \qquad (e^{-t}(e^{-t} x'(t))')' + 6[x(t)]^3 = 6(1 + 3e^{-t} + 3e^{-2t})$$

satisfies the conditions of Theorem 4. Thus no nonoscillatory solution of (12) approaches zero as $t \to \infty$. In fact, $x(t) = 1 + e^{-t}$ is a nonoscillatory solution of (12) which satisfies $\lim_{t\to\infty} x(t) = 1 \neq 0$.

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