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Cheh-Chih Yeh

A Fixed Point Theorem in Banach Spaces

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Analisi funzionale. — *A Fixed Point Theorem in Banach Spaces* (*). Nota di CHEH-CHIH YEH, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota è dimostrato un teorema di punto fisso in uno spazio di Banach X nel caso che una famiglia di trasformazioni di X di sè stessa.

Recently, Singh [5] proved a result on common fixed points in metric spaces. The purpose of this Note is to extend his result to Banach spaces. For related results, we refer to [1, 2, 3, 4].

THEOREM. *Let* Y *be a closed convex subset of a Banach space* X *and let* T_i $(i = 1, \cdots, n)$ *be a family of mappings of* Y *into* Y. *If* T_i $(i = 1, \cdots, n)$ *satisfies*

(*a*) $T_i T_j = T_j T_i$ *for* $i, j = 1, \cdots, n$,

(*b*) *there exist positive integers* $m_1, m_2, \cdots, m_n$ *such that*

$$\| T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n} x , T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n} y \|$$
$$\leq k \max \{\| x - y \| , \| x - T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n} x \| , \| y - T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n} y \| ,$$
$$\| y - T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n} x \| , \| x - T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n} y \|\}$$

for a $k \in (0, 1)$ *and for all* $x, y \in Y$. *Let* $x_0 \in Y$, $t \in (0, 1)$ *and* $x_{n+1} = (1 - t) x_n + t T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n} x_n$ *for each integer* $n \geq 0$. *If* $\{x_n\}_{n=0}^{\infty}$ *converges to a point* $u \in Y$, *then* T_i *has a common fixed point.*

Proof. Let $T = T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n}$. We first prove that u is the unique fixed point of T. Condition (*b*) implies

$$\| Tx - Ty \| \leq k \max \{\| x - y \| , \| x - Tx \| , \| y - Ty \| ,$$
$$\| x - Ty \| , \| y - Tx \|\} .$$

Since $x_{n+1} = (1 - t) x_n + t T x_n$,

$$\| x_{n+1} - Tu \| \leq (1 - t) \| x_n - Tu \| + t \| Tx_n - Tu \| . \tag{1}$$

It follows from (*b*) that

$$\| Tx_n - Tu \| \leq k \max \{\| x_n - Tx_n \| , \| u - Tu \| , \| Tu - x_n \| ,$$
$$\| Tx_n - u \| , \| x_n - u \|\} . \tag{2}$$

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From (1) and (2), we obtain

$$\| x_{n+1} - Tu \| \leq (1 - t) \| x_n - Tu \| + tk \max \{ \| Tx_n - x_n \| , \| Tu - u \| , \| Tu - x_n \| , \| Tx_n - u \| , \| x_n - u \| \} .$$

Since $\lim_{n\to\infty} x_n = u$, then $x_{n+1} - x_n = t(Tx_n - x_n) \to 0$ as $n \to \infty$. Thus $Tx_n - x_n$ tends to zero as $n \to \infty$. By (1)

$$\| u - Tu \| \leq (1 - t) \| u - Tu \| + kt \| u - Tu \| = [1 - (1 - k) t] \| u - Tu \| .$$

Since $1 - (1 - k) t \in (0 , 1)$, we have $u = Tu$.

Suppose there exists a point $v (\neq u)$ such that $Tv = v$. From *(b)*

$$\| u - v \| \leq k \| u - v \|$$

which is impossible since $k < 1$. Hence $u = v$.

Next, we prove that T_i has a common fixed point. By *(a)* and $T_i Tu = T_i u$, we have $TT_i u = T_i u$. Hence $T_i u$ is a fixed point of T. But T has a unique fixed point, hence $T_i u = u$ $(i = 1 , \cdots , n)$. Thus u is a common fixed point.

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