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A Remark on the Minimum Property

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Analisi funzionale. — *A Remark on the Minimum Property.* Nota di SIMEON REICH, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si dà una condizione sufficiente affinché certi sottoinsiemi di uno spazio di Banach godano di una proprietà qui detta di minimo.

Denote the closure and closed convex hull of a subset D of a Banach space E by cl (D) and clco (D) respectively, and let $\| D \| = \inf \{ |x| : x \in D\}$.

Recall that a closed subset F of E is said to possess the minimum property [3, p. 237] if it contains a point the norm of which equals $\|$ clco (F) $\|$. This concept has been useful in the study of the asymptotic behavior of nonexpansive mappings and semigroups [2, 3, 4–7].

A subset A of E×E is accretive if for each $[x_i , y_i] \in A$, $i = 1 , 2$, there exists $j \in J (x_1 - x_2)$ such that $(y_1 - y_2 , j) \geq 0$, where J is the (normalized) duality map from E to E*. We denote the domain and range of A by D (A) and R (A) respectively, and refer the reader to [7, Section 1] for other terms not defined here.

Suppose that A is accretive and closed, cl (D (A)) is a sunny non-expansive retract of E, and $R (I + rA) \supset$ cl (D (A)) for all $r > 0$. It is known [7, Propositions 2.14 and 2.16] that if E* is uniformly convex and its norm is Fréchet differentiable, then cl(R(A)) has the minimum property.

The purpose of this note is to prove that the uniform convexity of E* can be replaced with the weaker assumption that the norm of E is uniformly Gâteaux differentiable. The idea of the present proof is different from the previous one.

THEOREM 1. *Suppose that* $A \subset E \times E$ *is accretive,* cl (D (A)) *is a sunny nonexpansive retract of* E, *and* $R (I + rA) \supset$ cl (D (A)) *for all* $r > 0$. *If the norm of* E *is uniformly Gâteaux differentiable and the norm of* E* *is Fréchet differentiable, then* cl (R (A)) *has the minimum property.*

Proof. For positive r we restrict the resolvent and the Yosida approximation of A to cl (D (A)) and continue to denote them by J_r and A_r respectively. Let $S_r : [0 , \infty) \times$ cl (D (A)) $\to$ cl (D (A)) be the semigroup generated by $-Ar$, and let x be a point in cl(D(A)). By [1, Corrollary 4.3] the strong $\lim_{t \to \infty} dS_r (t , x)/dt$ exists and equals $-v_r$, where v_r is the element of least norm in cl (R (A_r)). Let $[u , w]$ belong to A. Since A is accretive, we have

$$(w + dS_r (t , x)/dt , \ J (u - J_r S_r (t , x))) \geq 0 .$$

(*) Nella seduta del 23 giugno 1977.

Since we also have

$$\lim_{t\to\infty} S_r(t, x)/t = \lim_{t\to\infty} (J_r S_r(t, x))/t = -v_r ,$$

we obtain $(w - v_r, J(v_r)) \geq 0$. Hence $(z - v_r, J(v_r)) \geq 0$ for all z in clco (R (A)). It follows that $|v_r| = \|\text{clco}(R(A))\|$. Since v_r (which turns out to be independent of r) belongs to cl (R (A)), this completes the proof.

Combining this result with [7, Theorem 2.8] we obtain the following improvement of [7, Theorem 2.17].

THEOREM 2. *Suppose that* $A \subset E \times E$ *is accretive*, cl (D (A)) *is a sunny nonexpansive retract of* E, *and* $R(I + rA) \supset \text{cl}(D(A))$ *for all* $r > 0$. *Let* S *be the semigroup generated by* $-A$ *on* cl (D (A)). *If the norm of* E *is uniformly Gâteaux differentiable and the norm of* E^* *is Fréchet differentiable, then for each* $x \in \text{cl}(D(A))$ *the strong* $\lim_{t\to\infty} S(t, x)/t = -v$, *where* v *is the element of least norm in* cl (R (A)).

In fact, v is also the element of least norm in cl $(R(A_r))$, cl $(R(I - S(1, \cdot)))$ and cl $(R(I - S_r(1, \cdot)))$.

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