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Luigi Mangiarotti

Neutron transport in a cylinder

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Fisica matematica. — *Neutron transport in a cylinder* (*). Nota di LUIGI MANGIAROTTI, presentata (**) dal Corrisp. G. SESTINI.

RIASSUNTO. — Attraverso un'analisi rigorosa del processo di diffusione di neutroni in un cilindro indefinito, si perviene a risultati di notevole significato fisico.

The purpose of this Note is to provide a rigorous mathematical analysis of the neutron transport problem in a homogeneous (infinite) cylinder with isotropic scattering in the one-velocity approximation. Our approach will be to use the integral equation for the neutron density (which using cylindrical coordinates is independent of z and of φ). In another paper (including also the proofs omitted or only sketched here), we will consider the case, of more relevant practical interest, in which the cylinder is surrounded by a reflector.

1. By making use of the optical unit $\alpha = \Sigma R > 0$ ($\Sigma > 0$ is the total macroscopic cross section for all processes of fission, scattering and absorption in the cylinder and R is its radius), the stationary neutron density $\rho_\alpha(r)$ must satisfy the linear integral equation [1]

$$\rho_\alpha(r) = c \int_0^1 T_\alpha(r, r')\, r'\, \rho_\alpha(r')\, dr', \tag{1}$$

where

$$4\pi T_\alpha(r, r') = \alpha \int_0^{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-\alpha(r^2+r'^2-2rr'\cos\varphi+z^2)^{1/2}}}{r^2+r'^2-2rr'\cos\varphi+z^2}\, d\varphi\, dz \tag{2}$$

and $c\ (\geq 1)$ is the average number of secondary neutrons per collision.

If $r \neq r'$, we easily see that

$$(r+r')\, T_\alpha(r, r') \leq \alpha K_0(\alpha\, |r-r'|), \tag{3}$$

where K_0 is the modified Bessel function (of zero order) [2]. Hence we get

$$T_\alpha(r, r')\, r' \leq \alpha K_0(\alpha\, |r-r'|). \tag{4}$$

Since $K_0(u) \sim -\ln u$ for small u, from (4) it follows that we can define (for any $\alpha > 0$) a linear integral operator T_α, whose kernel is just $T_\alpha(r, r')\, r'$, acting on the space $C(0, 1)$. Here $C(0, 1)$ is the space of all real valued functions defined and continuous on $[0, 1]$ endowed with the sup norm. It turns out that T_α is completely continuous (see [3], p. 162). Moreover, a simple

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computation shows also that T_α depends continuously on $\alpha \in (0, +\infty)$, that is, if $\beta \to \alpha$ then $\|T_\beta - T_\alpha\| \to 0$.

2. Let us now introduce the symmetric kernel

$$S_\alpha(r, r') = \sqrt{rr'}\, T_\alpha(r, r'). \tag{5}$$

From (3) it follows that

$$S_\alpha(r, r') \leq \frac{\alpha}{2} K_0(\alpha |r - r'|) \tag{6}$$

and hence we can define (for any $\alpha > 0$) a linear integral operator S_α, whose kernel is just $S_\alpha(r, r')$, acting on the Lebesgue space $L^2(0, 1)$. Note that the kernel $S_\alpha(r, r')$ is square integrable on $[0, 1] \times [0, 1]$. The operator S_α is symmetric and completely continuous. We have:

LEMMA 1. (i) S_α *depends continuously on* α, *that is, if* $\beta \to \alpha$ *then* $\|S_\beta - S_\alpha\| \to 0$.

(ii) S_α *is positive definite for any* $\alpha > 0$.

(iii) $S_\beta - S_\alpha$ *is positive definite if* $\beta > \alpha$.

Proof. The point (i) is a consequence of the continuity of $\int_0^1\int_0^1 S_\alpha^2(r, r')\, dr\, dr'$ with respect to the parameter α. Clearly this follows from the Lebesgue's dominated convergence theorem by recalling (6).

To prove (ii) and (iii), the crucial point is the following integral representation of $T_\alpha(r, r')$, that is

$$T_\alpha(r, r') = \alpha \int_0^{+\infty} \tan^{-1}(t/\alpha)\, J_0(rt)\, J_0(r't)\, dt, \tag{7}$$

where J_0 is the Bessel function (of zero order) [2]. A proof of (7) will not be given here. From (7) and (5), we get

$$(S_\alpha f, f) = \alpha \int_0^{+\infty} \tan^{-1}(t/\alpha)\, F^2(t)\, dt \quad , \quad f \in L^2(0, 1), \tag{8}$$

where F is the Hankel transform of f, that is

$$F(t) = \int_0^1 \sqrt{r} f(r)\, J_0(rt)\, dr, \qquad t \geq 0. \tag{9}$$

The change in the order of integrations in (8) can be justified by means of the Lebesgue's dominated convergence theorem Clearly, (ii) and (iii) follow now from (8).

3. As is well known [4], S_α has an infinite set of positive eigenvalues forming a sequence $\lambda_1(\alpha) \geq \lambda_2(\alpha) \geq \cdots$ converging to zero (zero is not an eigenvalue) and each eigenvalue is of finite multiplicity. The two following arguments show that the eigenfunctions of S_α are continuous. Indeed, from the inequality (6) we get

$$\int_0^1 S_\alpha^2(r, r')\, dr' \leq \frac{\alpha^2}{4} \int_{-\infty}^{+\infty} K_0^2(\alpha\, |r - r'|)\, dr' = \pi^2 \alpha/4, \tag{10}$$

for any $r \in [0, 1]$ hand hence S_α maps $L^2(0, 1)$ into its subspace of bounded functions. Moreover, S_α maps this subspace into that of continuous functions (see [3], p. 164). Then the result follows. Now, let φ_α be an eigenfunction of S_α. Since

$$\lim_{r \to 0+} \frac{1}{\sqrt{r}} \int_0^1 S_\alpha(r, r')\, \varphi_\alpha(r')\, dr'$$

exists and is finite, clearly the operators T_α and S_α both lead to the same eigenvalue problem.

4. In the following theorem we give some significant properties of the eigenvalues $\lambda_n(\alpha)$.

THEOREM 2. (i) *The eigenvalue* $\lambda_n(\alpha)$ *is a continuous and strictly increasing function of* α, $n \geq 1$.

(ii) *We have* $0 < \lambda_n(\alpha) < 1$ *and* $\lim_{\alpha \to 0+} \lambda_n(\alpha) = 0$, $n \geq 1$.

(iii) *The first eigenvalue* $\lambda_1(\alpha)$ *is simple and* $\lim_{\alpha \to +\infty} \lambda_1(\alpha) = 1$.

Proof. The continuity of $\lambda_n(\alpha)$ follows from the well known inequality $|\lambda_n(\beta) - \lambda_n(\alpha)| \leq \|S_\beta - S_\alpha\|$ and from point (i) of the Lemma 1. Moreover, from point (iii) of this Lemma we get $\lambda_n(\beta) > \lambda_n(\alpha)$ if $\beta > \alpha$ since then $S_\beta - S_\alpha$ is positive definite (see [4], p. 239).

To prove (ii), note that from the relation between the Hankel transform (9) and the two-dimensional Fourier transform and from the Plancherel theorem, we deduce that the function $\sqrt{t}\, F(t)$, $t \geq 0$, belongs to $L^2(0, +\infty)$ and that its norm is equal to $\|f\|$. Hence from (8) we get

$$(S_\alpha f, f) < \int_0^{+\infty} t F^2(t)\, dt = \|f\|^2, \qquad f \in L^2(0, 1),$$

and so $\|S_\alpha\| < 1$. On the other hand, from (10) it follows that $\|S_\alpha\| \leq \pi \sqrt{\alpha}/2$ and so $\lim_{\alpha \to 0+} \lambda_1(\alpha) = 0$.

Coming now to (iii), that $\lambda_1(\alpha)$ is simple follows at once from the well known Jentsch's theorem since $S_\alpha(r, r')$ is positive in $[0, 1] \times [0, 1]$, see

(2) and (5). Note that we may choose an eigenfunction corresponding to $\lambda_1(\alpha)$ such that is > 0 in $[0, 1]$. Finally, let $f(r) = \sqrt{2r}$, $r \in [0, 1]$. Then $f \in L^2(0, 1)$, $\|f\| = 1$ and from (9) we get $F(t) = \sqrt{2}\, J_1(t)/t$, since $\int_0^1 r J_0(rt)\, dr = J_1(t)/t$ (see [5], p. 22). Now, from the maximum property of $\lambda_1(\alpha)$ and from (8) we have

$$2\alpha \int_0^{+\infty} t^{-2} \tan^{-1}(t/\alpha)\, J_1^2(t)\, dt \leq \lambda_1(\alpha) < 1 .$$

Hence, $\lim_{\alpha \to +\infty} \lambda_1(\alpha) = 1$ since $\int_0^{+\infty} t^{-1} J_1^2(t)\, dt = 1/2$ (see [2], p. 403).

5. The main results concerning the solution of the integral equation (1) are summarized in the following theorem.

THEOREM 3. (i) *For any* $\alpha > 0$ *there is one and only one critical value* $c(\alpha) > 1$ *and one and only one neutron density* $\rho_\alpha \in C(0, 1)$ *such that* $\rho_\alpha(r) > 0$ *in* $[0, 1]$, $\|\rho_\alpha\| = 1$.

(ii) *The critical value* $c(\alpha)$ *is a continuous and strictly decreasing function of* α. *Moreover,* $\lim_{\alpha \to 0+} c(\alpha) = +\infty$ *and* $\lim_{\alpha \to +\infty} c(\alpha) = 1$.

(iii) *The neutron density* $\rho_\alpha \in C(0, 1)$ *depends continuously on* α, *that is, if* $\beta \to \alpha$ *then* $\|\rho_\beta - \rho_\alpha\| \to 0$.

Proof. Points (i) and (ii) follow from Theorem 2. Since $c(\alpha) = 1/\lambda_1(\alpha)$. Point (iii) can be proved by taking into account that T_α is a completely continuous operator which depends continuously on α and that $\lambda_1(\alpha)$ is simple (for more details see [5]).

To conclude, let us observe how all properties of the solution have a relevant physical meaning. Of particular interest are those of the critical value $c(\alpha)$. Note also that $\rho_\alpha(r)$ is a continuous function in $(0, +\infty) \times [0, 1]$.

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