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Fisica matematica. — *Quantization of a general system and application to the rigid sphere.* Nota II (*) di BRUNO CORDANI, presentata (**) dal Socio D. GRAFFI.

RIASSUNTO. — In questa seconda Nota, utilizzando i risultati della precedente, dimostriamo che il momento angolare di una sfera rigida è quantizzato secondo valori interi e semiinteri. Ricaviamo inoltre l'equazione di Pauli per spin qualsiasi. Dimostriamo infine che il limite della Lagrangiana della equazione di Pauli per alti numeri quantici è la Lagrangiana classica di un fluido con vorticità.

3. QUANTIZATION OF THE SPHERE

As an application of the previous section, let us consider the rigid sphere with a moment of inertia I and fixed center, and indicate with α, β, γ the Euler angles (the choice of Euler angles is agreement with [1, p. 5] if we put: $\alpha = \varphi_1, \beta = \vartheta, \gamma = \varphi_2$). The kinetic energy is

$$T = \frac{1}{2} I (\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2 + 2 \cos \beta \dot{\alpha} \dot{\gamma}) . \tag{3.1}$$

The Schrödinger equation for the steady states is

$$\frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \left(\sin \beta \frac{\partial \psi}{\partial \beta}\right) + \frac{1}{\sin^2 \beta} \left(\frac{\partial^2 \psi}{\partial \alpha^2} + \frac{\partial^2 \psi}{\partial \gamma^2} - 2 \cos \beta \frac{\partial^2 \psi}{\partial \alpha \partial \gamma}\right) + \frac{J^2}{\hbar^2} \psi = 0 , \tag{3.2}$$

$J = \sqrt{2 IE}$ being the angular momentum. We may write (3.2) in this form: $\hat{J}^2 \psi = J^2 \psi$ and therefore $\hat{J}$ is the operator of the total angular momentum. The eq. (3.2) is known in the theory of the Rotation Group as the equation of the Generalized Spherical Functions [1, p. 81]. The operator $\hat{J}^2$ may be written as: $\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2$, the $\hat{J}_k$'s being the operators of the projections of the angular momentum along the fixed axes: they are obtained from the classical expression through the usual replacement

$$p_\alpha \rightarrow \frac{\hbar}{i} \frac{\partial}{\partial \alpha} \quad ; \quad p_\beta \rightarrow \frac{\hbar}{i} \frac{\partial}{\partial \beta} \quad ; \quad p_\gamma \rightarrow \frac{\hbar}{i} \frac{\partial}{\partial \gamma} , \tag{3.3}$$

and satisfy the commutation rules

$$[\hat{J}_j , \hat{J}_h] = i\hbar \hat{J}_k \qquad (j , h , k \sim 1 , 2 , 3) . \tag{3.4}$$

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For the integration of (3.2) it is necessary to establish the periodicity conditions; so let us introduce the Cayley-Klein parameters

$$
\begin{aligned}
&X_1 = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2} \qquad && X_3 = \sin\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2} \\
&X_2 = \cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2} \qquad && X_4 = \sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2}\,.
\end{aligned}
\tag{3.5}
$$

They satisfy the relations

$$
\sum_1^4{}_i X_i^2 = 1 \quad ; \quad T = 2\,I \sum_1^4{}_i \dot{X}_i^2\,;
\tag{3.6}
$$

thanks to (3.6) the sphere is dynamically equivalent to a point constrained on the 3-dimensional sphere S^3: but, for the half-angles in (3.5), the points at opposite ends of a diameter denote the same configuration. As is well known, in the integration of the Schrödinger equation of a point constrained on S^2 (i.e. the equation of the Spherical Functions [1, p. 41]) one requires the solutions to be continuous and single valued, i.e. have period $2\pi/K$ (K integer). If we require the solutions of (3.2) to be continuous and single valued on S^3, we obtain, for the half-angles, that the period must be $4\pi/K$ (K integer).

Note that the source of this difference is topological: in fact the rotation group SO(3) is doubly connected while the S^2 sphere ($=SO(3)/SO(2)$) is simply connected. We have an analogous thing in Classical Mechanics; the density D, defined in (2.14), for a point constrained on S^2 is a function of $\sin^2\vartheta$ (ϑ being the colatitude) and so has period π, while for the rigid sphere is a function of $\cos\beta$ and so has period 2π. But ψ, being, roughly speaking, the square root of D, has therefore period 2π and 4π.

The solutions of (3.2) are [1, p. 85]

$$
T^{(l)}_{mn}(\alpha\,,\beta\,,\gamma) = e^{im\gamma}\,P^{(l)}_{mn}(\cos\beta)\,e^{in\alpha}
\tag{3.7}
$$

$$
\left(l = 0\,,\frac{1}{2}\,,1\,,\frac{3}{2}\cdots;m\,,n = l\,,l-1\cdots-l\right).
$$

The total angular momentum is quantized: $J = \hbar\sqrt{l(l-1)}$. Once l is fixed, $T^{(l)}$, a $(2l+1)\times(2l+1)$ square matrix, provides an irreducible representation with weight l of the Rotation Group. The action of the operator $\hat{J}_k$ on a function $T^{(l)}_{mn}$ transforms these in a linear combination of functions of the same representation. It is then possible introduce the square matrices $(\sigma^{(l)}_k)_{mn}$ with $m\,,n = l\,,l-1\cdots-l$:

$$
\hat{J}_k\,T^{(l)} = T^{(l)}\,\sigma^{(l)}_k,
\tag{3.8}
$$

where the usual matrix product is understood. The matrices $\sigma^{(l)}_k$ satisfy the same commutation rules (3.4): they are thus the infinitesimal generators of the Rotation Group. As is known, all entries are null except the following ones

$$(3.9)\qquad (\sigma_1^{(l)})_{m,m-1} = (\sigma_1^{(l)})_{m-1,m} = \frac{1}{2}\sqrt{(l+m)(l-m+1)}$$

$$(\sigma_2^{(l)})_{m,m-1} = -(\sigma_2^{(l)})_{m-1,m} = -\frac{i}{2}\sqrt{(l+m)(l-m+1)}$$

$$(\sigma_3^{(l)})_{m,m} = m \qquad (m = l, l-1 \cdots -l).$$

4. Charged sphere in an $e-m$ field (classical case)

Let us consider a sphere with mass m, moment of inertia I and charge q imbedded in an $e-m$ field, that we suppose slowly variable on intervals that are comparable with the sphere dimensions. The classical Hamiltonian is

$$(4.1)\qquad \mathrm{H} = \frac{1}{2m}\left(\mathbf{P} - \frac{q}{c}\mathbf{A}\right)^2 + q\Phi + \frac{1}{2\mathrm{I}}\left(\mathbf{J} - \mathrm{K}\frac{q\mathrm{I}}{2mc}\mathbf{B}\right)^2,$$

P being the linear momentum and **J** the angular momentum. K takes into account a possible different distribution of matter and charge density and is equal to 1 if they coincide. The Hamiltonian (4.1) is quadratic and linear in momenta and is therefore quantizable by means of the described method: it is interesting nevertheless to examine the classical system. The total angular momentum is a constant of the motion since $\{\mathrm{H}, \mathrm{J}^2\} = 0$. Moreover the Euler angle γ does not appear in (4.1) and therefore the conjugate momentum is another constant of the motion. Having found two constants of the motion, it is possible to express the angular part of H as a function of one coordinate $-\omega$ and its conjugate momentum ξ, since [2–3]

$$(4.2)\qquad \xi = \mathrm{J}_3 \quad ; \quad -\omega = -\operatorname{arctg}\left(\frac{\mathrm{J}_1}{\mathrm{J}_2}\right).$$

Neglecting the terms in J^2 and B^2 the Hamiltonian becomes

$$(4.3)\qquad \mathrm{H} = \frac{1}{2m}\left(\mathbf{P} - \frac{q}{c}\mathbf{A}\right)^2 + q\Phi - \mathrm{K}\frac{q}{2mc}[(\mathrm{J}^2 - \xi^2)^{1/2} \cdot$$

$$\cdot(\mathrm{B}_1 \cos\omega + \mathrm{B}_2 \sin\omega) + \xi \mathrm{B}_3].$$

From (4.3) we obtain an HJE whose complete integral is of the type $\mathrm{W}(x^k, t, \omega, \alpha_\mu)$, the α_μ's being four integration constants. This W is defined in the configuration space of (x, t, ω) but it is possible to give an equivalent description in which the configuration space is the usual space-time. This requires that, besides the W, also ξ and ω become some functions of space-time. Schiller [3] has given this: let us shortly summarize. Besides the α_μ's four other constants may be found by differentiating W:

$$(4.4)\qquad \beta_\mu = \frac{\partial \mathrm{W}}{\partial \alpha_\mu}.$$

Choose one of these equations, say $\beta_4 = \partial W/\partial\alpha_4$ and solve it for the variable ω. ω then becomes a function of $(x^k, t, \alpha_\mu, \beta_4)$. From the equation $\xi = -(\partial W/\partial\omega)$, ξ may also be found as a function of these same variables. Put

$$W(x^k, t, \omega(x^k, t, \alpha_\mu, \beta_4), \alpha_\mu) = S(x^k, t, \alpha_\mu, \beta_4), \tag{4.5}$$

and differentiate with respect to the time and the x's:

$$\frac{\partial W}{\partial t} - \xi\frac{\partial\omega}{\partial t} = \frac{\partial S}{\partial t} \quad ; \quad \frac{\partial W}{\partial x^k} - \xi\frac{\partial\omega}{\partial x^k} = \frac{\partial S}{\partial x^k}. \tag{4.6}$$

The HJE becomes

$$\frac{\partial S}{\partial t} + \xi\frac{\partial\omega}{\partial t} + \frac{1}{2m}\left(\nabla S + \xi\nabla\omega - \frac{q}{c}\mathbf{A}\right)^2 + q\Phi + H_{sp} = 0, \tag{4.7a}$$

where H_{sp} is the angular part of the Hamiltonian. To (4.7a) we can add the two Hamilton equations

$$\frac{\partial\xi}{\partial t} + \frac{1}{m}\nabla\xi\cdot\left(\nabla S + \xi\nabla\omega - \frac{q}{c}\mathbf{A}\right) = \frac{\partial H_{sp}}{\partial\omega} \tag{4.7b}$$

$$\frac{\partial\omega}{\partial t} + \frac{1}{m}\nabla\omega\cdot\left(\nabla S + \xi\nabla\omega - \frac{q}{c}\mathbf{A}\right) = -\frac{\partial H_{sp}}{\partial\xi}. \tag{4.7c}$$

Introduce the determinant

$$D = \det\left\|\frac{\partial}{\partial x^i}\left(\frac{\partial S}{\partial\alpha_k} + \xi\frac{\partial\omega}{\partial\alpha_k}\right)\right\|, \tag{4.8}$$

which satisfies the continuity equation

$$\frac{\partial D}{\partial t} + \nabla\cdot(D\boldsymbol{v}) = 0 \tag{4.7d}$$

where $\boldsymbol{v} = \frac{1}{m}\left(\nabla S + \xi\nabla\omega - \frac{q}{c}\mathbf{A}\right)$.

The solutions of the four equations (4.7) make stationary the Lagrangian density

$$\mathcal{L}_{cl} = D\left[\frac{\partial S}{\partial t} + \xi\frac{\partial\omega}{\partial t} + \frac{1}{2m}\left(\nabla S + \xi\nabla\omega - \frac{q}{c}\mathbf{A}\right)^2 + \right. \\ \left. + q\Phi - K\frac{q}{2mc}\mathbf{B}\cdot\mathbf{J}\right]. \tag{4.9}$$

S, ω, ξ are the Clebsch parameters since they were introduced by him studying the fluid vorticity: the eqs. (4.7) describe therefore a vortical fluid [4, p. 248].

See moreover the works of Schonberg [5] (where the Clebsch parameter are not yet tied with spin) and of Takabayasi–Vigier [6] (where their use is limited to the case: spin 1/2).

5. PAULI EQUATION AND ITS CLASSICAL LIMIT

To obtain the quantum equation of a charged sphere in an $e-m$ field it is enough to perform the replacement

$$\mathrm{P}_h \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x^h} \quad , \quad \mathrm{J}_k \rightarrow \hat{\mathrm{J}}_k \tag{5.1}$$

in the classical Hamiltonian (4.1), since $\sqrt{a}=1$. Usually one considers negligeable the terms in J^2 and B^2. So we obtain an equation in a 6-dimensional configuration space: it is physically more expressive to have an equation defined in the customary 3-dimensional space, utilizing the fact that we know the solution of the angular part. Putting.

$$\chi_m(x^k, \alpha, \beta, \gamma) = \sum_{-l}^{+l} \mathrm{T}_{mn}^{(l)} \psi_n(x) \tag{5.2}$$

or shortly

$$\chi = \mathrm{T}^{(l)} \psi , \tag{5.2'}$$

and substituting in the wave equation, we obtain, for the (3.8),

$$\mathrm{T}^{(l)} \left\{ \left[\frac{1}{2m} \left(\frac{\hbar}{i} \nabla - \frac{q}{c} \mathbf{A} \right)^2 + q\Phi - \mathrm{K} \frac{q\hbar}{2mc} \mathbf{B} \cdot \sigma^{(l)} - i\hbar \frac{\partial}{\partial t} \right] \psi \right\} = 0 . \tag{5.3}$$

Since (5.3) must be identically satisfied for whatever value of the Euler angles, it is equivalent to Pauli equation for any spin. For $l=1/2$ and $\mathrm{K}=2$ we have the electron equation. On account of what we have said on the classical limit of the quantum equations, the Pauli equation must have, as a limit for high quantum numbers, the classical equations of the vortical fluid, i.e. (4.7). Equivalently the quantum Lagrangian density

$$\begin{aligned} \mathscr{L}_q = \frac{1}{2m} \left(-\frac{\hbar}{i} \nabla\psi^+ - \frac{q}{c} \mathbf{A}\psi^+ \right) \cdot \left(\frac{\hbar}{i} \nabla\psi - \frac{q}{c} \mathbf{A}\psi \right) + q\Phi\psi^+ \psi + \\ - \mathrm{K} \frac{q\hbar}{2mc} \psi^+ \mathbf{B} \cdot \sigma^{(l)} \psi - i\hbar \frac{1}{2} \left(\psi^+ \frac{\partial\psi}{\partial t} - \frac{\partial\psi^+}{\partial t} \psi \right) \end{aligned} \tag{5.4}$$

must have, as a limit, the classical Lagrangian (4.9). It is instructive to check it directly: it is then necessary to find the equivalent, in our case, of the replacement

$$\psi = \mathrm{R}e^{(i/\hbar)\mathrm{S}}. \tag{5.5}$$

Having in mind the (3.7), we put

$$\psi_n(x,t) = \mathrm{R}(x,t)\, u_{ln}(\beta(x,t))\, e^{i\left(\frac{\mathrm{S}(x,t)}{\hbar} + n\omega(x,t)\right)}, \tag{5.6}$$

where

$$u_{ln}(\beta) = \mathrm{P}^{(l)}_{ln}(\cos\beta)\,. \tag{5.7}$$

The position (5.6) means to look for an orthonormal triad, varying with place and time, so that spin is directed along the z-axis. Substituting (5.6) in $\mathscr{L}_q$ we obtain e.g. for the first term

$$\begin{aligned}\hbar^2\,\nabla\psi^+\cdot\nabla\psi = \mathrm{R}^2 \Bigg[& \hbar^2 \left(\frac{\nabla \mathrm{R}}{\mathrm{R}}\right)^2 \sum_{+l}^{-l}{}_n\, \overset{*}{u}_{ln}\, u_{ln} + \\ & + (\nabla \mathrm{S})^2 \sum_{+l}^{-l}{}_n\, \overset{*}{u}_{ln}\, u_{ln} + 2\,\hbar\,\nabla \mathrm{S}\cdot\nabla\omega \sum_{+l}^{-l}{}_n\, n\overset{*}{u}_{ln}\, u_{ln} + \\ & + \hbar^2 (\nabla\omega)^2 \sum_{+l}^{-l}{}_n\, n^2\, \overset{*}{u}_{ln}\, u_{ln} + \hbar^2 (\nabla\beta)^2 \sum_{+l}^{-l}{}_n \frac{d\overset{*}{u}_{ln}}{d\beta}\,\frac{du_{ln}}{d\beta} + \\ & + \hbar^2\,\frac{\nabla \mathrm{R}}{\mathrm{R}}\cdot\nabla\beta \sum_{+l}^{-l}{}_n \left(u_{ln}\frac{d\overset{*}{u}_{ln}}{d\beta} + \overset{*}{u}_{ln}\frac{du_{ln}}{d\beta}\right) + \\ & + i\hbar\,\nabla\beta\cdot\nabla \mathrm{S} \sum_{+l}^{-l}{}_n \left(u_{ln}\frac{d\overset{*}{u}_{ln}}{d\beta} - \overset{*}{u}_{ln}\frac{du_{ln}}{d\beta}\right) + \\ & + i\hbar^2\,\nabla\beta\cdot\nabla\omega \sum_{+l}^{-l}{}_n\, n \left(u_{ln}\frac{d\overset{*}{u}_{ln}}{d\beta} - \overset{*}{u}_{ln}\frac{du_{ln}}{d\beta}\right)\Bigg]\,.\end{aligned} \tag{5.8}$$

(5.8) may be greately simplified. Since the representation $\mathrm{T}^{(l)}_{mn}$ is unitary, it follows that [1, p. 89]

$$\sum_{+l}^{-l}{}_n\, \overset{*}{u}_{ln}\, u_{ln} = 1\,. \tag{5.9}$$

The term $\Sigma_n\, n\overset{*}{u}_{ln}\, u_{ln}$ is equal to the element of the first row and first column of the matrix (we drop for simplicity the index l)

$$\mathrm{T}(0,\beta,0)\,\sigma_3\,\mathrm{T}^+(0,\beta,0) = \cos\beta\sigma_3 + \sin\beta\sigma_2 \tag{5.10}$$

and therefore, taking into account (3.9),

$$\sum_{+l}^{-l}{}_n\, n\overset{*}{u}_{ln}\, u_{ln} = l\cos\beta\,. \tag{5.11}$$

(5.10) is a particular case of the more general relation

$$\mathrm{T}\sigma_k\,\mathrm{T}^+ = \sum_{1}^{3}{}_h\, \mathrm{T}_{hk}\,\sigma_h \tag{5.12}$$

T_{hk} being the adjoint representation (i.e. the 3×3 representation). Since [1, p. 95]

$$\frac{du_{mn}}{d\beta} = \frac{n - m\cos\beta}{\sin\beta}\, u_{mn} - ic_m\, u_{m-1,n}, \tag{5.13}$$

where we have defined

$$c_m = \sqrt{(l+m)(l-m+1)}, \tag{5.14}$$

thanks to (5.9) and (5.11) we obtain

$$\sum_{+l}^{-l}{}_n \left(u_{ln} \frac{du_{ln}^*}{d\beta} + u_{ln}^* \frac{du_{ln}}{d\beta} \right) = 0. \tag{5.15}$$

Since [1, p. 87]

$$u_{mn}^* = (-1)^{m+n}\, u_{mn}, \tag{5.16}$$

we have on account of (5.13)

$$u_{ln}^* \frac{du_{ln}}{d\beta} - u_{ln} \frac{du_{ln}^*}{d\beta} = -ic_m\,(u_{mn}^*\, u_{m-1,n} + u_{mn}\, u_{m-1,n}^*) = 0. \tag{5.17}$$

Since the term $\Sigma_n n^2 u_{mn}^* u_{mn}$ is equal to the element of the m-th row and m-th column of the matrix

$$T(0,\beta,0)\,\sigma_3\sigma_3\,T^+(0,\beta,0) = (T\sigma_3 T^+)(T\sigma_3 T^+), \tag{5.18}$$

from (5.10) and (3.9) we have

$$\sum_{+l}^{-l}{}_n\, n^2 u_{mn}^* u_{mn} = m^2\cos^2\beta + \frac{1}{2}\sin^2\beta\,[l(l+1) - m^2]. \tag{5.19}$$

From (5.13) and (5.19) we have

$$\sum_{+l}^{-l}{}_n \frac{du_{mn}^*}{d\beta}\frac{du_{mn}}{d\beta} = \frac{1}{2}[l(l+1) - m^2]. \tag{5.20}$$

Since for high values of l

$$l \simeq \sqrt{l(l+1)}, \tag{5.21}$$

$l\hbar$ is equal to total angular momentum, so that

$$l\hbar\cos\beta = \xi. \tag{5.22}$$

At last we have

$$\hbar^2\,\nabla\psi^+\cdot\nabla\psi = R^2 \left\{ \hbar^2 \left(\frac{\nabla R}{R}\right)^2 + (\nabla S + \xi\nabla\omega)^2 + \right. \tag{5.23}$$
$$\left. + \frac{1}{2}\hbar^2\,[(\nabla\beta)^2 + \sin^2\beta\,(\nabla\omega)^2]\,[l(l+1) - l^2] \right\}.$$

Obviously the classical limit is

$$\hbar^2 \nabla\psi^+ \cdot \nabla\psi \to \mathrm{D}\,(\nabla \mathrm{S} + \xi \nabla\omega)^2. \tag{5.24}$$

Analogously we may complete the demonstration: $\mathscr{L}_q \to \mathscr{L}_{cl}$. It is easy moreover to prove that the limit of the quantum density current

$$\boldsymbol{j}_q = \frac{1}{2\,m} \sum_{+l}^{-l}{}_n \left[\psi_n^* \left(\frac{\hbar}{i} \nabla - \frac{q}{c} \mathbf{A} \right) \psi_n + \psi_n \left(- \frac{\hbar}{i} - \frac{q}{c} \mathbf{A} \right) \psi_n^* \right] \tag{5.25}$$

is the classical density current

$$\boldsymbol{j}_c = \mathrm{D}\, \frac{1}{m} \left(\nabla \mathrm{S} + \xi \nabla\omega - \frac{q}{c} \mathbf{A} \right) \tag{5.26}$$

in accordance with the previous definition of $\boldsymbol{v}$.

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