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On the non-degenerate critical manifolds

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Geometria differenziale. — *On the non-degenerate critical manifolds*. Nota di MONIR S. MORSY, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si dimostra il teorema enunciato nell'Introduzione.

INTRODUCTION

Let M be a compact, connected, smooth n-manifold and let $f : M \to R$ be a smooth real-valued function on M. By M^a we mean the set $f^{-1}(-\infty, a]$.

This paper will start out with some preliminaries (§ 1), then we will give the proof of the following theorem (§ 2).

If f has two compact non-degenerate critical manifolds N , L *such that f takes the minimum values* 0 *on* N *and the maximum values* 1 *on* L, *then* $\exists$ *an* ε *small enough such that*:

(1) $M^{\varepsilon} = f^{-1}[0, \varepsilon] \simeq$ *Normal disc bundle of* N *in* M ,

(2) $M^{1-\varepsilon} = f^{-1}[1-\varepsilon, 1] \simeq$ *Normal disc bundle of* L *in* M ,

(3) $M - M^{1-\varepsilon} \simeq$ *Normal vector bundle of* N ,

(4) $M - M^{\varepsilon} \simeq$ *Normal vector bundle of* L.

§ 1 PRELIMINARIES [see 1 or 2]

DEFINITION 1. A point $x \in M$ is called a critical point of f if the induced map:

$$df: \quad T_x(M) \to T_{f(x)}(R) \simeq R$$

is zero, where $T_x(M)$ denotes the tangent space of M at a point $x \in M$. The real number $f(x)$ is called a critical value of f.

DEFINITION 2. If x is a critical point of f, we define a symmetric bilinear functional Hf_x, called the Hessian of f at x, on $T_x(M)$ by

$$Hf_x(v, w) = \tilde{v}_x(\tilde{w}(f)) \qquad \text{for} \quad v, w \in T_x(M),$$

whre $\tilde{v}$, $\tilde{w}$ are the extensions of v, w respectively, to vector fields and $\tilde{w}(f)$ means the directional derivative of f along $\tilde{w}$.

DEFINITION 3. The space $\{v \in T_x(M) : Hf_x(v, w) = 0 \ \forall w \in T_x(M)\}$ is called the null space of Hf_x.

(*) Nella seduta del 14 maggio 1977.

DEFINITION 4. The compact connected submanifold V of M will be called a non-degenerate critical manifold of f, if

(1) every point $x \in \mathrm{V}$ is a critical point of f;

(2) $\forall$ points $x \in \mathrm{V}$ the null space of $\mathrm{H}f_x$ is precisely the tangent space of V.

§ 2. PROOF OF THE THEOREM

Let us first introduce a Riemann metric on M as follows: let $\{\mathrm{U}_1, \cdots, \mathrm{U}_n\}$ be a finite open covering of M with coordinate systems, and $\{\mathrm{P}_{\mathrm{U}_1}, \cdots, \mathrm{P}_{\mathrm{U}_n}\}$ a partition of unity corresponding to $\{\mathrm{U}_1, \cdots, \mathrm{U}_n\}$. Let $\{x_1^{\mathrm{U}_i}, \cdots, x_n^{\mathrm{U}_i}\}$ be coordinates in U_i and $x \in \mathrm{U}_i$. For every pair $(v, w) \in \mathrm{T}_x(\mathrm{M})$ we define $\mathrm{P}_{\mathrm{U}_i}(v, w)$ by setting the inner product

$$\langle \partial/\partial x_\alpha^{\mathrm{U}_i}, \partial/\partial x_\beta^{\mathrm{U}_i} \rangle = \delta_{\alpha\beta}$$

and extending by linearity

$$\langle \tilde{v}, \tilde{w} \rangle = \sum_{i=1}^{n} \mathrm{P}_{\mathrm{U}_i}(v, w)$$

defines a global Riemann metric on M. This metric enables us to define a special vector field ∇f on M, the gradient of f, by

$$\langle f, v \rangle = (\tilde{v}(f)) .$$

Obviously, $\nabla f = 0$ is equivalent to $\mathrm{d}f(x) = 0$.

Proof of 1. Let $x \in \mathrm{N}$ and let g be the restriction of f to the geodesic plane perpendicular to N in x. ∇g is transversal to an ε-sphere, since $\mathrm{H}g_x$ is non-degenerate by definition (4). From the compactness of N we deduce the existence of a δ-neighborhood $\nu_\delta(\mathrm{N})$, with respect to an invariant metric, of N in M such that Δf is transversal to $\partial\nu_\delta(\mathrm{N})$ (the boundary of $\nu_\delta(\mathrm{N})$). Since $\partial\nu_\delta(\mathrm{N})$ is an orbit, it follows that $f =$ constant say ε on $\partial\nu_\delta(\mathrm{N})$. But since f has no critical points outside N (see [2]), it follows that ε is the maximum value of f on ν_δ and $f \neq \varepsilon$ at the points of $\nu_0(\mathrm{N}) - \partial\nu_\delta(\mathrm{N})$. Also ε is similarly the minimum value of f and $\mathrm{M} - \nu_\delta(\mathrm{N})$ and $f \neq \varepsilon$ at the interior points of $\mathrm{M} - \nu_\delta(\mathrm{N})$. Thus $\nu_\delta(\mathrm{N}) \approx \mathrm{M}^\varepsilon$.

Proof of 2. Replaing f by $1 - f$ and N by L in (1) we get the result.

Proof of 3. From (1) and (2), M has the decomposition:

$$\mathrm{M} = \mathrm{M}^\varepsilon \cup \mathrm{B} \cup \mathrm{M}^{1-\varepsilon},$$

where $\mathrm{B} = [0, 1] \times \partial\mathrm{M}^\varepsilon = [0, 1] \times \partial\mathrm{M}^{1-\varepsilon}$. Since $\mathrm{M}^{1-\varepsilon}$ is a normal disc bundle of L, we have

$$\mathrm{M}^{1-\varepsilon} - \mathrm{L} \simeq \partial\mathrm{M}^{1-\varepsilon} \times [0, 1]$$

and so

$$M - M^{1-\varepsilon} = M^{\varepsilon} \cup B \cup (M^{1-\varepsilon} - L) \simeq M^{\varepsilon} - \partial M^{\varepsilon}$$

$$\simeq \text{ normal vector bundle of } N.$$

Proof of 4. It is proved in a similar fashion as (3) with N , L interchanged.

REFERENCES

[1] R. BOTT and H. SAMELSON (1958) – *Application of the theory of Morse to symmetric spaces*, « American Journ. of Math. », *80* 964–1029.
[2] J. MILNOR (1963) – *Morse Theory*, « Annals of Mathematics studies », *51*.