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## A Remark on the Tangent Bundle $T(M_n)$ with $g^M$ over a Symmetric Riemann Manifold $M_n$

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**Geometria differenziale.** — *A Remark on the Tangent Bundle* $T(M_n)$ *with* $g^M$ *over a Symmetric Riemann Manifold* $M_n$. Nota di TANJIRO OKUBO, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Designato con $T(M_n)$ il fascio tangente di una varietà riemanniana $M_n$ dotato della metrica $g^M$ di Sasaki-Muto, si dimostra che, dal fatto che $M_n$ sia simmetrica nel senso di E. Cartan, non segue in generale la simmetria di $T(M_n)$.

## INTRODUCTION

Some years ago K. Yano and the present author developed the tensor calculus on the tangent bundle $T(M_n)$ over a Riemannian manifold $M_n$ by endowing the so-called Sasaki-Muto metric $g^M$, [6], and the paper was followed by a trial of giving the geometrical significance to those functions, vector and tensor fields explicitly by establishing the structural equations along the two complementary distributions defined at each point of $T(M_n)$, [4]. In both papers we showed that there does not exist in $T(M_n)$ a space of non-vanishing constant curvature, and this implies specifically that $T(M_n)$ over a Riemannian manifold $M_n$ of non-vanishing constant curvature cannot be a space of constant curvature. Then the question arises, when the base manifold $M_n$ is symmetric in the sense of E. Cartan, if $T(M_n)$ must also be symmetric.

Since the curvature tensor of the Riemann connection $\nabla^M$ with respect to the metric $g^M$ have the sixteen components in each $\pi^{-1}\{U(x^i)\}$, $U(x^i)$ being the local coordinate neighbourhood of $M_n$ (see § 1), the actual computation of taking the covariant differentiation of them with respect to $\nabla^M$ which actually has the eight components $\Gamma^{\alpha}_{\gamma\beta}$ is tremendously cumbersome and is almost impossible. In this paper we present the following theorem on this matter, without making this tedious work, which states:

THEOREM 1. *Let* $M_n$ *be any Riemann manifold which is symmetric in the sense of E. Cartan. Then its tangent bundle* $T(M_n)$ *with* $g^M$ *is in general not symmetric.*

§ 1 is a brief introduction of the structure of a tangent bundle with $g^M$, and in § 2 we prove the theorem by using the results on symmetric space due to A. Lichnerowicz [1] and K. Nomizu [2].

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## § 1. TANGENT BUNDLE $T(M_n)$ OVER A RIEMANNIAN MANIFOLD $M_n$ WITH THE METRIC $g^M$

Let $M_n$ be a Riemann manifold covered by a system of coordinate neighbourhoods $\{U ; (x^i)\}$ (1) and let $\nabla$ be its Riemannian connection. Then $\nabla$ is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) + \\ + g([Z, X], Y) + g(X, [Z, Y])$$

where $g$ is the Riemann metric of M and X, Y and Z are arbitrary vector fields on M. It has in $\{U ; (x^i)\}$ the local expression

$$\left\{ {h \atop j\ i} \right\} = \frac{1}{2} g^{hk} (\partial_j g_{ki} + \partial_i g_{jk} - \partial_k g_{ji}),$$

which is the Christoffel symbol, where $g_{ji}$ are the components of $g$ in $\{U ; (x^i)\}$ and $\partial_j = \partial / \partial x^j$.

Let $T(M_n)$ be tangent bundle over $M_n$. Denoting by $\pi$ the projection $T(M_n) \to M_n$, we introduce at each point $x^A(x^i, y^i)$ of $\pi^{-1}(U)$, the two complementary distributions spanned by $\delta_\alpha = (\delta_i, \delta_{n+1})$:

$$(1) \qquad \delta_i = \partial_i - \Gamma_i^j \partial_{n+1} \quad , \quad \delta_{n+i} = \partial_{n+i} = \partial / \partial y^i,$$

where $y^i = x^{n+i}$ are the components of a tangent vector defined at each point $\pi(x^A)$ of the base manifold and $\Gamma_i^j = \left\{ {j \atop i\ l} \right\} y^l$. We call $\delta_A$ the adapted frame at $x^A$ in $T(M_n)$ and it has the components $\Lambda_\alpha^{\ A} = (\Lambda_i^{\ A}, \Lambda_{n+i}^{\ \ A})$;

$$\Lambda_i^{\ A} = (\delta_i^h, -\Gamma_i^h) \quad , \quad \Lambda_{n+i}^{\ \ A} = (0, \delta_{n+i}^{\ \ h})$$

with respect to the natural base $\partial_A = (\partial_i, \partial_{n+1})$ in $\pi^{-1}(U)$ and $\delta_i$ is right invariant by the action of any element of the structure group $o(n)$ of $T(M)$, [3] The coframe $\delta x^B$ dual to the adapted frame given by $\langle \delta x^B, \delta_A \rangle \delta_B^A$ has the expression

$$\delta x^\beta = \Lambda^\beta{}_A \, dx^B$$

in $\pi^{-1}(U)$, where

$$\Lambda_\alpha^{\ A} \Lambda^\beta{}_A = \delta_\alpha^\beta \quad , \quad \Lambda_\alpha^{\ A} \Lambda^\alpha{}_B = \delta_B^{\ A},$$

$$\Lambda^i{}_A = (\delta_h^i, 0) \quad , \quad \Lambda^{n+i}{}_A = (\Gamma_h^i, \delta_h^i).$$

(1) We make the following convention for the indices:
The capital Roman letters A, B, C, ··· and the small greek letters $\alpha, \beta, \gamma, \delta, \varepsilon, \cdots$ run over the range $1, 2, \cdots, n, n+1, \cdots, 2n$, and the small Roman letters $a, b, c, d, e, h, i, j, k, \cdots$ over the range $1, 2, \cdots, n$.

From this we see that $\delta x^{B}$ is composed of two parts

$$\delta x^{i} = \mathrm{d}x^{i} \quad , \quad \delta x^{n+i} = \mathrm{d}y^{i} + \Gamma^{i}_{j}\, \mathrm{d}x^{j}.$$

$\delta_{i}$ and $\delta_{\bar{i}}$ are called the basis of the horizontal and vertical vectors at $x^{A}$ in $T(M_{n})$ respectively, and $\delta x^{i}$ and $\delta x^{n+i}$ are called the basis of the horizontal and vertical one-forms at $x^{A}$ in $T(M_{n})$ respectively. The non-holonomic object of these two distributions is given by

$$\Omega_{\beta\alpha}{}^{\gamma} = -\,\Omega_{\alpha\beta}{}^{\gamma} = \Lambda^{\gamma}{}_{A}\,(\delta_{\beta}\,\Lambda_{\alpha}{}^{A} - \delta_{\alpha}\,\Lambda_{\beta}{}^{A})$$

and for the various range of indices $\alpha$, $\beta$ and $\gamma$, $\Omega_{\beta\alpha}{}^{\gamma}$ are found to be

$$\Omega_{ji}{}^{n+h} = -\,\Omega_{ji}{}^{n+h} = -\,K_{jil}{}^{h}\, y^{l},$$
$$\Omega_{n+ji}{}^{h} = -\,\Omega_{in+j}{}^{h} = -\,\Gamma^{h}_{ij},$$

all other $\Omega$'s are zero, where $K_{jil}{}^{h}$ are the components in $\{U;(x^{i})\}$ of the curvature tensor $K$ of $\nabla$ given by

$$K(X,Y)Z = \nabla_{X}\nabla_{Y}Z - \nabla_{Y}\nabla_{X}Z - \nabla_{[X,Y]}Z.$$

We introduce in $T(M_{n})$ the so-called Muto-Sasaki Riemann metric $g^{M}$ defined by [6]

$$\mathrm{d}\bar{s}^{2} = g_{ji}\,\mathrm{d}x^{j}\,\mathrm{d}x^{i} + g_{ji}\,\delta y^{i}\,\delta y^{i},$$

and with $g^{M}$ we endow $T(M_{n})$ with the unique Riemann connection $\nabla^{M}$ given by

$$2g^{M}(\nabla^{M}{}_{\mathbf{X}}\mathbf{Y},\mathbf{Z}) = \mathbf{X}g^{M}(\mathbf{Y},\mathbf{Z}) + \mathbf{Y}g^{M}(\mathbf{Z},\mathbf{X}) - \mathbf{Z}g(\mathbf{X},\mathbf{Y})$$
$$+\, g^{M}([\mathbf{X},\mathbf{Y}],\mathbf{Z}) + g^{M}([\mathbf{Z},\mathbf{X}],\mathbf{Y}) + g^{M}(\mathbf{X},[\mathbf{Z},\mathbf{Y}])$$

with respect to the natural base $\partial_{A}$, where $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are arbitrary vector fields in $T(M_{n})$. Its components with respect to the adapted frame are given by

$$\Gamma^{\alpha}_{\gamma\beta} = \tfrac{1}{2}\, g^{\alpha\varepsilon}(\delta_{\gamma}\, g_{\beta\varepsilon} + \delta_{\beta}\, g_{\varepsilon\alpha} - \delta_{\varepsilon}\, g_{\alpha\beta}) + \tfrac{1}{2}(\Omega_{\delta\beta}{}^{\alpha} + \Omega^{\alpha}{}_{\gamma\beta} + \Omega^{\alpha}{}_{\beta\gamma})$$

where

$$\Omega^{\alpha}{}_{\gamma\beta} = g^{\alpha\delta}\, g_{\beta\varepsilon}\,\Omega_{\delta\gamma}{}^{\varepsilon},$$

and $g_{\alpha\beta}$ are the components of $g^{M}$, that is,

$$(g_{\alpha\beta}) = \begin{bmatrix} g_{ji} & 0 \\ 0 & g_{ji} \end{bmatrix}$$

and $g_{\alpha\beta}\, g^{\beta\gamma} = \delta^{\gamma}_{\beta}$. For the various ranges of the indices, it turns out to be

$$'\Gamma^h_{ji} = \left\{ {h \atop j\ i} \right\} \;,\; '\Gamma^{n+h}_{ji} = -\tfrac{1}{2}\, K_{jil}{}^h\, y^l \;,\; '\Gamma^h_{n+j\,i} = -\tfrac{1}{2}\, K_{jli}{}^h\, y^l \;,\; \Gamma^{n+h}_{n+j\,i} = 0\,,$$

$$'\Gamma^h_{ji} = 0 \quad,\quad '\Gamma_{ji} = \left\{ {h \atop j\ i} \right\} \quad,\quad '\Gamma^h_{ji} = 0 \quad,\quad \Gamma^h_{ji} = 0\,.$$

The curvature tensor $K^M$ of $\nabla^M$ is defined in the adapted frame by, [4],

$$\nabla_{\mathbf{U}}{}^M \nabla_{\mathbf{V}}{}^M \mathbf{W} - \nabla_{\mathbf{V}}{}^M \nabla^M{}_{\mathbf{U}} \mathbf{W} - \nabla^M$$
$$\nabla^M_{\mathbf{U}} \mathbf{V} - \nabla^M_{\mathbf{V}} \mathbf{U} - \Omega(\mathbf{U},\mathbf{V})^{\mathbf{W}} = K^M(\mathbf{U},\mathbf{V})\,\mathbf{W}$$

where $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ are arbitrary vector fields in $T(M_n)$ and $\nabla^M{}_{\mathbf{U}}\mathbf{V}$ has in $\pi^{-1}(U)$ the local expression

(2) $$\nabla^M_U V = U^\alpha (\delta_\alpha V^\beta + \Gamma^{\ \beta}_{\alpha\gamma} V^\gamma)\, \delta_\beta\,.$$

The components $\tilde{K}^M{}_{\delta\gamma\beta}$ of $K^M$ are given by

$$'K_{kji}{}^h = K_{kji}{}^h + \tfrac{1}{4}(K_{dck}{}^h K_{jib}{}^d - K_{dcj}{}^h K_{kib}{}^d - 2\, K_{fci}{}^h K_{kjb}{}^f)\, y^b y^c\,,$$

$$'K_{n+k\,ji}{}^h = \tfrac{1}{2}(\nabla_j K_{kai}{}^h)\, y^a\,,$$

$$'K_{k\,n+j\,i}{}^h = -\tfrac{1}{2}(\nabla_k K_{j0i}{}^h)\, y^a\,,$$

$$'K_{kj\,n+i}{}^h = -\tfrac{1}{2}(\nabla_k K_{iaj}{}^h - \nabla_j K_{iak}{}^h)\, y^a\,,$$

$$'K_{n+k\,n+j\,i}{}^h = K_{kji}{}^h + \tfrac{1}{4}(K_{kca}{}^h K_{jbi}{}^a - K_{jca}{}^h K_{kbi}{}^a)\, y^c y^b\,,$$

$$'K_{k\,n+j\,n+i}{}^h = \tfrac{1}{2} K_{kij}{}^h - \tfrac{1}{4} K_{jca}{}^j K_{kbi}{}^a\, y^c y^b\,,$$

$$'K_{n+k\,n+j\,n+i}{}^h = \tfrac{1}{2} K_{ijk}{}^h - \tfrac{1}{4} K_{jca}{}^h\, y^c y^b\,,$$

$$'K_{kij}{}^{n+h} = 0 \quad,\quad \tilde{K}_{n+kji}{}^{n+h} = \tfrac{1}{2}(\nabla_i K_{kja}{}^h)\, y^a\,,$$

$$'K_{n+k\,ji}{}^{n+h} = -\tfrac{1}{2} K_{jik}{}^h - \tfrac{1}{4} K_{jac}{}^h K_{kbi}{}^a\, y^c y^b\,,$$

$$'K_{k\,n+j\,i}{}^{n+h} = \tfrac{1}{2} K_{kji}{}^h + \tfrac{1}{4} K_{kac}{}^h K_{jbi}{}^a\, y^c y^b\,,$$

$$'K_{kj\,n+i}{}^{n+h} = K_{kji}{}^h + \tfrac{1}{4}(K_{kac}{}^h K_{ibj}{}^a - K_{jac}{}^h K_{ibk}{}^a)\, y^b\,,$$

$$'K_{n+k\,n+j\,i}{}^{n+h} = 0\;,\; 'K_{n+k\,j\,n+i}{}^{n+h} = 0\;,\; 'K_{k\,n+j\,n+i}{}^{n+h} = 0\;,\; 'K_{n+k\,n+j\,n+i}{}^{n+h} = 0.$$

Then the components $K^M{}_{\gamma\beta} = K^M{}_{\delta\gamma\beta}{}^\delta$ of the Ricci curvature tensor are found to be

$$'K_{ji} = K_{ji} - \tfrac{1}{4}(K^a{}_{jc}{}^d K_{aibd} + 2\, K^a{}_{ic} K_{ajbd} + K_{jac}{}^d K^a{}_{ibd})\, y^c y^b\,,$$

$$'K_{n+j\,i} = -\tfrac{1}{2}(\nabla_j K_{ai} - \nabla_a K_{ji})\, y^a\,,$$

$$'K_{j\,n+i} = -\tfrac{1}{2}(\nabla_i K_{aj} - \nabla_a K_{ij})\, y^a\,,$$

$$'K_{n+j\,n+i} = \tfrac{1}{4} K^{ca}{}_{cj} K_{cabi}\, y^c y^b\,,$$

from which we find that the scalar curvature $[K^M] = g^{\beta\alpha} K^M_{\beta\alpha}$ takes the form

$$\tilde{K} = K - \frac{1}{4} K^{ea}{}_c{}^d K_{eabd} y^c y^b \tag{3}$$

where $K_{ji}$ and $K$ are the components in $\{U ; (x^i)\}$ of the Ricci curvature and the scalar curvature of $\nabla$ in $M_n$.

## § 2. Proof of the theorem

Let us suppose that $T(M_n)$ is symmetric in the sense of E. Cartan, that is,

$$\nabla^M{}_\varepsilon \tilde{K}_{\delta\gamma\beta}{}^\alpha = 0 .$$

Since $\nabla^M$ is Riemannian, we have

$$\nabla^M_\varepsilon \tilde{K}_{\gamma\beta} = 0$$

and hence

$$\nabla^M_\varepsilon \tilde{K} = \delta_\varepsilon \tilde{K} = 0 . \tag{4}$$

If we take $n + j$ for $\varepsilon$ and use (3), we have

$$K^{ea}{}_c{}^d K_{eabd} y^c = 0 \tag{5}$$

in virtue of the second of the operators defined by (1) and of the fact that the components $K_{kji}{}^h$ of the curvature tensor of $\nabla$ on $M_n$ and the scalar curvature $K$ do not depend upon the $y$'s Applying again $\delta_{n+j}$ to (5), we have

$$K^{ea}{}_c{}^d K_{eabe} = 0 , \tag{6}$$

and multiplying $g^{cb} g^{ed}$, we have

$$K^{khji} K_{khji} = 0 . \tag{7}$$

Because of (6), the scalar curvature $K$ of $T(M_n)$ given in (3) takes the form

$$\tilde{K} = K$$

and on taking $i$ for $\varepsilon$ in (4), we have $\partial_i K = 0$, that is,

$$K = a \quad (a\text{: const.}) . \tag{8}$$

(7) and (8) are the necessary conditions for a $M_n$ so that the tangent bundle $T(M_n)$ with $g^M$ may be a symmetric space.

We now assume that the base manifold $M_n$ is symmetric in the sense of E. Cartan with respect to $\nabla$ too, that is,

$$\nabla_l K_{kji}{}^h = 0 . \tag{9}$$

Then we have as above

$$(10) \qquad \nabla_l K_{ji} = 0 ,$$

and the equations

$$H_{kjipq}{}^{h} = K_{kjs}{}^{h} K_{ipq}{}^{s} - K_{kji}{}^{s} K_{spq}{}^{h} - K_{kjp}{}^{s} K_{isq}{}^{h} - K_{kjq}{}^{s} K_{ips}{}^{h} = 0$$

as the integrability condition of (9). Since we do not impose any topological condition on $M_n$, we suppose that $M_n$ is compact and orientable. For this case A. Lichnerowicz [1] proved that if $M_n$ satisfies the conditions (10) and (11), it must be symmetric in the sense of E. Cartan and for this case one gets

$$(12) \qquad K^{kjih} K_{kjih} = C \qquad (C: \text{const.}) \text{ (2)} .$$

Generalizing this theorem, K. Namizu [2] proved that, if an irreducible Riemann manifold $M_n$ (not necessarily compact and orientable) admits a transitive group of motions whose linear isotropy group at any point contains the homogeneous holonomy group at that point, the manifold $M_n$ is symmetric and (12) holds (3). On the other hand, we cannot expect that the constant C in (12) is always zero for any symmetric space. For example, let $M_n$, $n \geq 2$, be a non-flat Riemannian manifold of constant curvature, i.e.

$$K_{kjih} = k\,(g_{kh}\, g_{ji} - g_{ki}\, g_{jh}) , \qquad (k = \text{const}, \neq 0);$$

the

$$K^{kjih} K_{kjih} = 2\, k^2\, n\,(n - 1) = \text{const} \neq 0 .$$

But, in order that $T(M_n)$ may be symmetric, we found in (7) that the constant C in (12) should vanish all the time, which we cannot expect for all the symmetric spaces $M_n$'s, that is, the tangent bundle $T(M_n)$ with $g^M$ over a symmetric Riemann manifold $M_n$ has not necessarily to be a symmetric space. *Q.E.D.*

Let us suppose that the base manifold $M_n$ is non-flat Kaehlerian with the complex dimension $n = 2m$ and that $M_n$ can be isometrically imbedded in an $(n + 1)$-dimensional flat Kaehler space $K_{n+1}$ as an invariant hypersurface in the sense that the complex structure F of $K_{n+1}$ keeps the tangent plane of the imbedded Kaehler manifold invariant at each point and the almost complex $f$ of the hypersurface induced from F coincides with the complex structure of $M_n$. Then it has been proved by the present author [3] that, the condition for $M_n$ to be the case, its curvature tensor of $\nabla$ should satisfy

$$(13) \qquad K^{kjih} K_{kjih} = K^2 , \qquad K < 0$$

(2) See also, K. Yano [5], p. 223.
(3) See, K. Yano [5], p. 224.

where $K_{kji}{}^{h}$ are the components of the curvature tensor in the form of real representation. Thus, taking account of (7), we can state

THEOREM 2. *If* $M_n$ *is a Kaehlerian manifold and* $T(M_n)$ *with* $g^M$ *is a symmetric space, then* $M_n$ *cannot be isometrically imbedded in a flat Kaehler space* $K_{n+1}$ *as an invariant hypersurface unless it is locally flat.*

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