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On a method of successive approximations

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Analisi funzionale. — *On a method of successive approximations* (*).
Nota di SILVIO MASSA, presentata (**) dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Sia T una mappa quasi non espansiva che applica in sè un sottoinsieme chiuso e convesso di uno spazio di Banach strettamente convesso, e sia $U = \sum_{i=0}^{\infty} \alpha_i T^i$, $\alpha_i \geqq 0$, $\Sigma\alpha_i = 1$, $\alpha_k \cdot \alpha_{k+1} \neq 0$ per almeno un k intero. In questa Nota si dimostra che T ed U hanno gli stessi punti fissi e si dànno condizioni affinchè $\{U^n(x)\}$ converga ad un punto fisso. Nel caso X uniformemente convesso e T non espansiva con almeno un punto fisso, si dimostra che U è asintoticamente regolare.

1. INTRODUCTION

Here and throughout the paper, let X be a strictly convex Banach space, K a closed and convex subset of X, T a quasi-nonexpansive selfmapping of K, F (T) the set of the fixed points of T (1).

In a recent Paper we considered the mapping

$$(1.1) \qquad S = \sum_{i=0}^{\infty} \alpha_i T^i , \qquad \alpha_i \geqq 0 , \qquad \Sigma\alpha_i = 1 , \qquad \alpha_0 > 0 , \qquad \alpha_1 > 0$$

and we proved that S and T have the same fixed points; that $\{S^n(x_0)\}$ converges to a fixed point if there exists a converging subsequence of it and T is continuous; finally that, if X is uniformly convex, then S is asymptotically regular.

In this Paper we consider the mappings (more general than S)

$$(1.2) \qquad U = \sum_{i=0}^{\infty} \alpha_i T^i , \qquad \alpha_i \geqq 0 , \qquad \Sigma\alpha_i = 1 , \qquad \alpha_k \cdot \alpha_{k+1} \neq 0$$

for at least an integer k.

We prove that, for these mappings, the above properties still hold; the asymptotic regularity of U is proved under the more restrictive condition that T is nonexpansive.

Further results on the convergence of $\{U^n(x)\}$ for mappings T of particular kind can be found in [2].

(*) Lavoro eseguito nell'ambito del G.N.A.F.A.

(**) Nella seduta del 14 maggio 1977.

(1) We recall that T is said quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|T(x) - p\| \leqq \|x - p\| \; \forall p \in F(T) \wedge \forall x \in K$. In this Paper we consider only mappings that have at least one fixed point: in this case (in accordance with the previous definition) every nonexpansive map is quasi-nonexpansive. Many Authors studied quasi-nonexpansive mappings; references on principal results can be found in [3], [4] and [1].

2. RESULTS

THEOREM 1. $F(T) = F(U)$.

LEMMA 1. U *is quasi-contractive* (2).

THEOREM 2. *Let* T *be continuous and* $\exists x_0 \in K$ *such that* $\{U^n(x_0)\}$ *has a convergent subsequence. Then* $\{U^n(x_0)\}$ *converges to a fixed point of* T (3).

THEOREM 3. *Let* X *be uniformly convex and* T *nonexpansive. Then* U *is asymptotically regular.*

REMARKS. 1) The assumption $\alpha_k \cdot \alpha_{k+1} \neq 0$ cannot be removed (4) neither it can be replaced by the weaker assumption: "there are h and j such that $\alpha_h \cdot \alpha_j \neq 0$" (5), even with the additional condition h even and j odd.

Indeed let $X = K = C$, $T : x \to e^{2\pi i/3} x$.

For $\alpha_0 = \alpha_3 = \frac{1}{2}$ we have $U = \frac{1}{2} I + \frac{1}{2} T^3 = I$: Theorems 1 and 2 fail.

For $\alpha_1 = \alpha_4 = \frac{1}{2}$ we have $U = \frac{1}{2} T + \frac{1}{2} T^4 = T$: Lemma 1 and Theorem 3 fail.

2) If T has no fixed point, we can neither ensure that U is defined nor, if U is defined, that it is asymptotically regular. Indeed, if $X = K = R$, $T : x \to x + 1$ it would be

$$U(x) = \sum_{i=0}^{\infty} \alpha_i T^i(x) = \sum_{i=0}^{\infty} \alpha_i (x + i) = x + \sum_{i=0}^{\infty} i \cdot \alpha_i$$

and it is not sure that

$$\sum_{i=0}^{\infty} i\alpha_i \in R \quad \left(\text{e.g. } \alpha_0 = 0 , \alpha_i = \frac{1}{i(i+1)}\right) {}^{(6)} .$$

3. PROOFS.

We can suppose, without any loss of generality, that $0 \in F(T)$. Thus

$$\| T^i(x) \| \leqq \| x \| \qquad \forall x \in K \wedge \forall i .$$

(2) i.e. $F(U) \neq \varnothing$ and $\| U(x) - p \| < \| x - p \| \; \forall p \in F(T) \wedge \forall x \in K - F(T)$.

(3) See [1], Remark 5.

(4) Exception made for Theorem 1, in the trivial case $\alpha_1 = 1$.

(5) See also [1], Remark 3.

(6) It is possible that U is asymptotically regular even if T has no fixed point. Indeed let $X = R$, $K = [1, +\infty)$, $T : x \mapsto x + 1/x$. T and $U = \frac{1}{2}(I + T)$ are asymptotically regular.

THEOREM 1. $x = T(x) \Rightarrow x = U(x)$ obvious.
Let

$$x = U(x) = \Sigma \alpha_i T^i(x).$$

We have

$$\| x \| \leqq \Sigma \alpha_i \| T^i(x) \| \leqq \| x \|,$$

therefore

$$\| T^i(x) \| = \| x \| \qquad \forall i : \alpha_i \neq 0$$

and, from the strict convexity of X [7]

$$T^i(x) = x \qquad \forall i : \alpha_i \neq 0.$$

In particular

$$x = T^k(x) = T^{k+1}(x) = T(T^k(x)) = T(x).$$

LEMMA 1. Let $p \in F(T) = F(U)$, $x \in K - F(T)$.
Suppose that

$$\| U(x) - p \| = \| x - p \|.$$

Thus

$$\| x - p \| = \| \Sigma \alpha_i (T^i(x) - p) \| \leqq \Sigma \alpha_i \| T^i(x) - p \| \leqq \| x - p \|,$$

therefore

$$x - p = T^k(x) - p = T^{k+1}(x) - p$$

that implies $x = T(x)$. Absurd.

THEOREM 2. The proof is as in Theorem 2 of [1].

THEOREM 3. Let $x_{n+1} = U(x_n)$. If $x_n \notin F(T)$, the sequence $\{ \| x_n \| \}$ is strictly decreasing. Let $d = \lim \| x_n \|$ and suppose $d > 0$ (if $d = 0$ then $\{x_n\}$ converges). We have, for every h such that $\alpha_h \neq 0$

$$\lim_n \| x_{n+1} - T^h(x_n) \| = 0. \tag{3.1}$$

Indeed

$$\limsup_{n \to \infty} \| T^h(x_n) \| \leqq d \quad , \quad \limsup_{n \to \infty} \left\| \sum_{i \neq h} \frac{\alpha_i}{1 - \alpha_h} T^i(x_n) \right\| \leqq d,$$

$$\| x_{n+1} \| = \left\| \alpha_h T^h(x_n) + (1 - \alpha_h) \sum_{i \neq h} \frac{\alpha_i}{1 - \alpha_h} T^i(x_n) \right\| \to d$$

and (3.1) follows from uniform convexity of X.

(7) See [1].

As

$$\| U^{n+1}(x) - U^n(x) \| = \| x_{n+1} - x_n \| \leqq \| x_{n+1} - T^k(x_n) \| + \| T^k(x_n) - x_n \| ,$$

Theorem 3 is an immediate consequence of (3.1) and of the following

LEMMA 2. $\lim_{n \to \infty} \| x_{n+1} - T^j(x_{n+1}) \| = 0 \qquad \forall j \geqq 1.$

a) $j = 1$.

$$\| x_{n+1} - T(x_{n+1}) \| \leqq \| x_{n+1} - T^{k+1}(x_n) \| + \| T^{k+1}(x_n) - T(x_{n+1}) \| \leqq$$
$$\leqq \| x_{n+1} - T^{k+1}(x_n) \| + \| x_{n+1} - T^k(x_n) \| \to 0 \quad \text{from (3.1)}.$$

b) $j > 1$.

$$\| x_{n+1} - T^j(x_{n+1}) \| \leqq \sum_{i=1}^{j} \| T^{i-1}(x_{n+1}) - T^i(x_{n+1}) \| \leqq$$
$$\leqq j \| x_{n+1} - T(x_{n+1}) \| \to 0 \quad \text{from } a).$$

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