Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Evangelos Livieratos

Connection in the actual gravity field of the Earth in terms of the disturbing potential

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **62** (1977), n.4, p. 523–528.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1977_8_62_4_523_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1977.

Geodesia. — *Connection in the actual gravity field of the Earth in terms of the disturbing potential.* Nota di EVANGELOS LIVIERATOS (*), presentata (**) dal Socio A. MARUSSI.

RIASSUNTO. — Nella presente Nota si ottengono i simboli di Christoffel di seconda specie riferiti al campo attuale di gravità terrestre come somma dei corrispondenti simboli riferiti ad un campo normale e delle « connessioni di perturbazione » causate dal potenziale anomalo terrestre.

LIST OF SYMBOLS

Φ , Λ	astronomic latitude and longitude;
W	actual potential of the Earth;
φ , λ	normal latitude and longitude;
U	normal potential;
x^i	intrinsic coordinates ($x^1 = \Phi , x^2 = \Lambda , x^3 = W$);
y^i	normal coordinates ($y^1 = \varphi , y^2 = \lambda , y^3 = U$);
θ^i	coordinate disturbancies ($\theta^1 = \xi , \theta^2 = \varepsilon , \theta^3 = T$);
$\boldsymbol{v}_i^*$	covariant vector frame referred to the actual field.
$\boldsymbol{v}_i$	covariant vector frame referred to the normal field;
$\boldsymbol{v}^{i*}$	contravariant vector frame referred to the actual field.
$\boldsymbol{v}^i$	contravariant vector frame referred to the normal field.
$\xi , \eta = \varepsilon \cos \varphi$	deflection of the vertical components, in the meridian and in the parallel respectively.
T	disturbing potential;
$\boldsymbol{i}_k^* \equiv \boldsymbol{i}^{*k}$	astronomic orthonormal base: $\boldsymbol{i}_1^*$, $\boldsymbol{i}_2^*$ towards the actual North and East respectively; $\boldsymbol{i}_3^*$ towards the actual Zenith. (clockwise);
$\boldsymbol{i}_k \equiv \boldsymbol{i}^k$	normal orthonormal base: $\boldsymbol{i}_1$, $\boldsymbol{i}_2$ towards the normal north and east respectively; $\boldsymbol{i}_3$ towards the normal zenith. (clockwise).
ρ , N	radius of curvature of the normal meridian and of prime vertical section respectively;
γ , g	normal gravity and actual gravity respectively;
δ_i^j	Kronecker's symbol;
$\left\{ {h \atop r\ s} \right\}^*$	Christoffel's symbols of the second kind referred to the actual gravity field of the Earth.
$\left\{ {h \atop r\ s} \right\}$	Christoffel's symbols of the second kind referred to the normal field;
$\mathcal{D}_{rs}^h$	connection disturbancies;
g_{ij}^*	components of the metric tensor concerning the actual field.
g^{*ij}	reciprocal components of the metric tensor;
$f = (\lg \gamma)_\varphi = \dfrac{\partial \lg \gamma}{\partial \varphi}$.	
M_j^i	a 3×3 matrix with i rows and j columns.
$\theta_{y^j}^i , \theta_{y^j y^k}^i$	partial derivatives $\dfrac{\partial \theta^i}{\partial y^j} , \dfrac{\partial^2 \theta^i}{\partial y^j \partial y^k}$.

(*) Technical University, Athens.

(**) Nella seduta del 16 aprile 1977.

Dealing with problems in Intrinsic Geodesy (Marussi, 1949, 1951, 1952, 1976), it is many times necessary to connect, or displace local vectors or local reference vector-frames in the actual gravity field of the Earth. In the present paper we derive the Christoffel's symbols of the second kind referred to the actual gravity field of the Earth, as a sum of the corresponding Christoffel's symbols of the second kind referred to a model gravity field (e.g. a Somigliana-Pizzetti type of model field or a model field derived, for instance, from a satellite solution), and of the "connection disturbancies" caused by the disturbing potential of the Earth.

1. Any point P in the actual 3-dimensional space can be described by an ordered set of intrinsic coordinates x^i and/or an ordered set of normal coordinates y^j, where and from now on all upper and lower indices run from 1 to 3 and the index summation convention holds. By definition $x^h = y^h + \theta^h$, where θ^h are quantities related between them, as it will be shown.

The elementary displacement dP of P in terms of x^i and y^j is given by the relations $dP = dx^i \boldsymbol{v}_i^* = dy^j \boldsymbol{v}_j$; the x^i and y^j being scalar quantities, have gradients given by $\boldsymbol{v}^{*s} = \text{grad } x^s$ and $\boldsymbol{v}^t = \text{grad } y^t$.

Recalling the relation between x^i, y^j, θ^k we obtain:

$$\boldsymbol{v}^{*i} = \boldsymbol{v}^i + \text{grad } \theta^i \tag{1.1}$$

Referring the vector-triad $\boldsymbol{v}^j$ to the base $\boldsymbol{i}_s$ we have the relation:

$$\boldsymbol{v}^j = \mathrm{A}_j^s \, \boldsymbol{i}_s \tag{1.2}$$

where the matrix-elements A_j^s are given by Marussi 1952, p. 83, eq. V-27 for a Somigliana-Pizzetti type of normal field. The matrix could be as well computed for any normal field, e.g. for a field derived from a satellite solution given in terms of spherical harmonics.

Furthermore the following relation holds:

$$\text{grad } \theta^k = \mathrm{B}_t^k \, \boldsymbol{v}^t \tag{1.3}$$

where B_t^k are the elements of the Jacobian matrix $\partial(\theta^1, \theta^2, \theta^3)/\partial(y^1, y^2, y^3)$.

2. Introducing (1.2) into (1.3) and ignoring the second order terms in the quantities θ^i and f, we obtain the relation:

$$\text{grad } \theta^i = \mathrm{B}_t^i \mathrm{A}_s^t \, \boldsymbol{i}^s = \mathrm{C}_s^i \, \boldsymbol{i}^s. \tag{2.1}$$

Combining (1.1), (1.2) and (2.1):

$$\boldsymbol{v}^{*i} = \mathrm{A}_s^i \, \boldsymbol{i}^s + \mathrm{C}_s^i \, \boldsymbol{i}^s = \mathrm{D}_s^i \, \boldsymbol{i}^s \tag{2.2}$$

the elements of the matrix D being:

$$D_1^1 = \frac{1}{\rho}(1+\xi_\varphi) \quad ; \quad D_2^1 = \frac{\sec\varphi}{N}\xi_\lambda \quad ; \quad D_3^1 = -\gamma\xi_v + \frac{f}{\rho}$$

$$D_1^2 = \frac{1}{\rho}\varepsilon_\varphi \quad ; \quad D_2^2 = \frac{\sec\varphi}{N}(1+\varepsilon_\lambda) \quad ; \quad D_3^2 = -\gamma\varepsilon_v$$

$$D_1^3 = \frac{1}{\rho}T_\varphi \quad ; \quad D_2^3 = \frac{\sec\varphi}{N}T_\lambda \quad ; \quad D_3^3 = -\gamma(1+T_v).$$

Noting that in the first approximation the determinant of D is always non-zero, (Livieratos 1976), we can invert the matrix D obtaining $\boldsymbol{v}_i^*$:

(2.3) $$\boldsymbol{v}_i^* = E_s^i \, \boldsymbol{i}^s.$$

(2.2) and (2.3) are satisfying the condition $\boldsymbol{v}^{*i} \times \boldsymbol{v}_j^* = \delta_j^i$, with the help of which we derive the elements of the matrix E:

$$E_1^1 = \rho(1-\xi_\varphi) \quad ; \quad E_2^1 = -N\varepsilon_\varphi\cos\varphi \quad ; \quad E_3^1 = \frac{1}{\gamma}T_\varphi$$

$$E_1^2 = -\rho\xi_\lambda \quad ; \quad E_2^2 = N\cos\varphi(1-\varepsilon_\lambda) \quad ; \quad E_3^2 = \frac{1}{\gamma}T_\lambda$$

$$E_1^3 = \frac{f}{\gamma} - \rho\xi_v \quad ; \quad E_3^2 = -N\varepsilon_v\cos\varphi \quad ; \quad E_3^3 = -\frac{1}{\gamma}(1-T_v).$$

3. The transformation of the $\boldsymbol{i}_s$ into $\boldsymbol{i}_h^*$ base is given by the rotational relation:

$$\boldsymbol{i}_s^* = R_h^s \, \boldsymbol{i}^h$$

the elements of the orthonormal matrix R being, (see for instance Livieratos 1976, Appendix 1):

$$R_1^1 = R_2^2 = R_3^3 = 1 \quad ; \quad R_2^1 = -R_1^2 = -\varepsilon\sin\varphi \quad ; \quad R_3^1 = -R_1^3 = -\xi \, ;$$

$$R_3^2 = -R_2^3 = -\varepsilon\cos\varphi$$

Obviously:

(3.1) $$\boldsymbol{i}_h = R_s^h \, \boldsymbol{i}^{*s}.$$

Introducing (3.1) into (2.2), we obtain:

(3.2) $$\boldsymbol{v}^{*i} = D_s^i R_k^s \, \boldsymbol{i}^{*k} = M_h^i \, \boldsymbol{i}^{*h}$$

the elements of the matrix M being:

$$M_1^1 = \frac{1}{\rho}(1+\xi_\varphi) \quad ; \quad M_2^1 = \frac{\varepsilon\sin\varphi}{\rho} + \frac{\sec\varphi}{N}\xi_\lambda \quad ; \quad M_3^1 = \frac{\xi}{\rho} + \frac{f}{\rho} - \gamma\xi_v$$

(3.3.1) $$M_1^2 = \frac{1}{\rho}\varepsilon_\varphi - \frac{\varepsilon\,\mathrm{tg}\,\varphi}{N} \quad ; \quad M_2^2 = \frac{\sec\varphi}{N}(1+\varepsilon_\lambda) \quad ; \quad M_3^2 = \frac{\varepsilon}{N} - \gamma\varepsilon_v$$

$$M_1^3 = \frac{1}{\rho}T_\varphi + \gamma\xi \quad ; \quad M_2^3 = \frac{\sec\varphi}{N}T_\lambda + \varepsilon\gamma\cos\varphi \quad ; \quad M_3^3 = -\gamma(1+T_v).$$

The matrix M_t^s is the first order approximation in therms of the disturbing quantities, of the corresponding general matrix $\mathcal{M}_t^s$ given by Marussi 1951, p. 27, eq. 22.3, the elements of which, written in terms of the North and East normal curvatures $(\varkappa_\eta^N, \varkappa_\eta^E)$ of the North and East geodetic torsions (τ_g^N, τ_g^E) and of the components of the curvature of the line of force (k_1, k_2), are the following:

$$\begin{aligned} &\mathcal{M}_1^1 = \varkappa_\eta^N \quad ; \quad \mathcal{M}_2^1 = \tau_g^N \quad ; \quad \mathcal{M}_3^1 = k_1 \\ (3.3.2) \qquad &\mathcal{M}_1^2 = -\tau_g^E \sec\Phi \quad ; \quad \mathcal{M}_2^2 = \varkappa_\eta^E \sec\Phi \quad ; \quad \mathcal{M}_3^2 = k_2 \sec\Phi \\ &\mathcal{M}_1^3 = \mathcal{M}_2^3 = 0 \quad ; \quad \mathcal{M}_3^3 = -g \end{aligned}$$

where $\mathcal{M}_t^s = M_t^s + 0_2$ and $\tau_g^N = -\tau_g^E$.

Comparing the elements of $\mathcal{M}$ and M we obtain the relations:

$$\varkappa_\eta^N \cong \frac{1}{\rho}(1+\xi_\varphi) \quad ; \quad \varkappa_\eta^E \cong \frac{1}{N}(1+\varepsilon_\lambda) \; ;$$

$$k_1 \cong \frac{f}{\rho} + \frac{\xi}{\rho} - \gamma\xi_v \; ; \quad k_2 \cong \left(\frac{\varepsilon}{N} - \gamma\varepsilon_v\right)\cos\varphi$$

$$\tau_g^N \cong \left(\frac{1}{\rho}\varepsilon_\varphi - \frac{\varepsilon}{N}\operatorname{tg}\varphi\right)\cos\varphi$$

and

$$(3.4) \qquad T_\varphi \cong -\gamma\rho\xi \quad ; \quad T_\lambda \cong -\gamma N\varepsilon\cos^2\varphi \quad ; \quad \xi_\lambda \cong (\cos\varphi)_\varphi \,.$$

The equations (3.4) show that the quantities θ^i are not independent, the last equation being the Villarceau condition. These equations verify the conditions derived by Marussi (1973, p. 5, 6), using another approach.

4. From the relations (2.2), (2.3) and (3.1) we can easily compute the metric of the actual gravity field g_{ij}^* as well as its reciprocal elements g^{*ij}, ignoring second order terms:

$$g_{11}^* = \rho^2(1-2\xi_\varphi) \quad ; \quad g_{12}^* = g_{21}^* = -\rho^2\xi_\lambda + N^2\varepsilon_\varphi\cos^2\varphi$$

$$g_{13}^* = g_{31}^* = \frac{\rho f}{\gamma} - \rho^2\xi_v - \frac{1}{\gamma^2}T_\varphi \quad ; \quad g_{22}^* = N^2\cos^2\varphi(1-2\varepsilon_\lambda)$$

$$g_{23}^* = g_{23}^* = -N^2\varepsilon_v\cos^2\varphi - \frac{1}{\gamma^2}T_\lambda \quad ; \quad g_{33}^* = \frac{1}{\gamma^2}(1-2T_v)$$

$$g^{*11} = \frac{1}{\rho^2}(1+2\xi_\varphi) \quad ; \quad g^{*12} = g^{*21} = \frac{1}{\rho^2}\varepsilon_\varphi + \frac{\sec^2\varphi}{N^2}\xi_\lambda$$

$$g^{*13} = g^{*31} = -\frac{\gamma}{\rho}(f+\xi) + \gamma^2\xi_v \quad ; \quad g^{*22} = \frac{\sec^2\varphi}{N^2}(1+2\varepsilon_\lambda)$$

$$g^{*23} = g^{*32} = -\frac{\gamma}{N}\varepsilon + \gamma^2\varepsilon_v \quad ; \quad g^{*33} = \gamma^2(1+2T_v)\,.$$

5. The Christoffel's symbols of the second kind, concerning the actual gravity field, can be written in the following form:

$$\left\{ {h \atop r \quad s} \right\}^* = \left\{ {h \atop r \quad s} \right\} + \mathscr{D}^h_{rs} \tag{4.1}$$

whre $\mathscr{D}^h_{rs}$ are the " connection disturbancies " which added to the Christoffel's symbols of the second kind referred to any normal field, provide the corresponding symbols of the actual field. The $\left\{ {h \atop r \quad s} \right\}$ referred to a Somigliana–Pizzetti ellipsoidal type of normal field are given by Marussi (1952, p. 82, eq. V–24).

Here we derive the first approximation of the " connection disturbances " $\mathscr{D}^h_{rs}$ in terms of the deflections of the vertical and of the disturbing potential:

$$\mathscr{D}^1_{11} = -\xi + \gamma\rho\xi_v - \xi_{\varphi\varphi}$$

$$\mathscr{D}^1_{12} = \mathscr{D}^1_{21} = -\frac{N}{\rho}(\varepsilon + \varepsilon_\varphi \operatorname{tg}\varphi)\cos^2\varphi - \frac{\rho}{N}\xi_\lambda \operatorname{tg}\varphi - \xi_{\varphi\lambda}$$

$$\mathscr{D}^1_{13} = \mathscr{D}^1_{31} = \frac{1}{\gamma\rho}T_v - \xi_{\varphi v}$$

$$\mathscr{D}^1_{22} = -\frac{N}{\rho}[\varepsilon\cos\varphi - (\xi_\varphi - 2\varepsilon_\lambda)\operatorname{tg}\varphi - \gamma\rho\xi_v]\cos^2\varphi - \xi_{\lambda\lambda}$$

$$\mathscr{D}^1_{23} = \mathscr{D}^1_{32} = -\frac{N}{\rho}\varepsilon_v\cos\varphi\sin\varphi - \frac{1}{\gamma}\left(\frac{1}{N} - \frac{1}{\rho}\right)\xi_\lambda - \xi_{\lambda v}$$

$$\mathscr{D}^1_{33} = \frac{2}{\gamma\rho}\xi_v - \xi_{vv}$$

$$\mathscr{D}^2_{11} = \gamma\rho\,\varepsilon_v + \frac{\varepsilon\operatorname{tg}\varphi}{N}\rho_\varphi - \varepsilon_{\varphi\varphi}$$

$$\mathscr{D}^2_{12} = \mathscr{D}^2_{21} = -\frac{\rho}{N}(\xi - \xi_\varphi\operatorname{tg}\varphi) - \varepsilon_{\varphi\lambda}$$

$$\mathscr{D}^2_{13} = \mathscr{D}^2_{31} = \frac{\rho}{N}\varepsilon_v\operatorname{tg}\varphi - \frac{1}{\gamma}\left[\frac{1}{\rho} - \frac{1}{N}(1 + f\operatorname{tg}\varphi)\right]\varepsilon_\varphi - \varepsilon_{\varphi v}$$

$$\mathscr{D}^2_{22} = -\left(\varepsilon - \frac{N}{\rho}\varepsilon_\varphi\operatorname{tg}\varphi - \gamma N\varepsilon_v\right)\cos^2\varphi - \varepsilon_{\lambda\lambda}$$

$$\mathscr{D}^2_{23} = \mathscr{D}^2_{32} = \frac{\rho}{N}\xi_v\operatorname{tg}\varphi + \frac{1}{\gamma N}T_v - \varepsilon_{\lambda v}$$

$$\mathscr{D}^2_{33} = \frac{1}{\gamma}\left[\frac{1}{N} + (1 + f\operatorname{tg}\varphi)\right]\varepsilon_v + \frac{\varepsilon}{\gamma N}f_v\operatorname{tg}\varphi - \varepsilon_{vv}$$

$$\mathscr{D}^3_{11} = \gamma\rho T_v - T_{\varphi\varphi}$$

$$\mathscr{D}^3_{12} = \mathscr{D}^3_{21} = \gamma(\rho\varepsilon\operatorname{tg}\varphi - N\varepsilon_\varphi)\cos^2\varphi - \gamma\rho\xi_\lambda - T_{\varphi\lambda}$$

$$\mathscr{D}^3_{13} = \mathscr{D}^3_{31} = 0 = \xi - \gamma\rho\xi_v - T_{\varphi v}$$

$$\mathscr{D}^3_{22} = -\gamma N(\xi\operatorname{tg}\varphi - 2\varepsilon_\lambda - T_v)\cos^2\varphi - T_{\lambda\lambda}$$

$$\mathscr{D}^3_{23} = \mathscr{D}^3_{32} = 0 = (\varepsilon - \gamma N\varepsilon_v)\cos^2\varphi - T_{\lambda v} \quad ; \quad \mathscr{D}^3_{33} = -T_{vv}.$$

Acknowledgements. This work has been carried out at the Institute of Geodesy and Geophysics, University of Trieste. I am grateful to Prof. A. Marussi who introduced me in the world of Intrinsic Geodesy, as well as for his valuable suggestions and discussions.

Thanks are due to Dr. F. Bocchio. University of Trieste, for his comments.

The financial support for my stay in Italy by the "Commissione Geodetica Italiana", is kindly acknowledged.

REFERENCES

LIVIERATOS E. (1976) – *On the geodetic singularity problem*, «Manuscr. Geod.», *4*, W. Berlin.

MARUSSI A. (1949) – *Fondements de géométrie differentielle absolue du champ potential terrestre*, « Bull. Geod. », *14*, Paris.

MARUSSI A. (1951) – *Fondamenti di geodesia intrinseca*, « Pubbl. Comm. Geod. Ital. », III ser. No. 7, Milano.

MARUSSI A. (1952) – *Intrinsic geodesy*. Map. Chart. Lab., Ohio St. Univ., Techn. Rep. No. 159, Columbus.

MARUSSI A. (1973) – *On the three dimensional computation of geodetic networks and anomalies* « Rend. Accad. Naz. Lincei », ser. VIII, *55* (5), Roma.

MARUSSI A. (1976) – *Intrinsic geodesy*, « Boll. Geod. Sc. Aff. », *1*, Firenze.