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Quantization of a general system and application to the rigid sphere

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Fisica matematica. — *Quantization of a general system and application to the rigid sphere.* Nota I [(*)] di BRUNO CORDANI, presentata [(**)] dal Socio D. GRAFFI.

RIASSUNTO. — In questa prima Nota, per mezzo di uno schema generale delle teorie fisiche, diamo una semplice descrizione del processo di quantizzazione di un generico sistema meccanico non stazionario. Tenendo presente un risultato di Van Vleck e Schiller, mostriamo che si può risalire dal caso limite della Meccanica Classica al caso generale della Meccanica Ondulatoria senza ambiguità. In questo modo l'usuale interpretazione probabilistica della Meccanica Quantistica nasce spontaneamente.

1. INTRODUCTION

It is a common statement that spin of elementary particles has not classical analogous: but this is not correct, as some authors have proved [1–4]. The quoted authors have in effect demonstrated that it is possible to quantize the model of the rigid sphere through a suitable substitution, in the classical Hamiltonian, of differential operators instead of canonical momenta. But there are even some points that, in our opinion, require explanations.

1) This quantization method is somewhat formal: historically Schrödinger deduced his equation remarking the analogy between the eikonal equation of geometrical optics and Hamilton-Jacobi equation (HJE) and considering the classical mechanics (CM) like the limit of a new wave mechanics (WM). This way appears more natural although Rot [5] has remarked that the way back from CM to WM in the standard procedure, is not unambigousy. We prove instead that, taking into account a result of Van Vleck [6] and Schiller [7], the Correspondence Principle allows us to obtain unambigously the wave equation of a general system. This rielaboration of well known facts finds an effective settlement thanks to a classification scheme of physical theories that Tonti has recently proposed [8–9] (sec. 2).

2) The probabilistic interpretation is postulated: we show instead that, thanks to the quoted result, it is spontaneously suggested from the quantization method.

3) The angular momentum of the quantized sphere takes up integer and half-integer values whilst the angular momentum in the Kepler motion of a point takes up only integer values. In the standard development this difference is in substance postulated: we show instead in sec. 3 that it can be deduced.

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In the other two sections we show the connection between this quantization method and the properties of the "spin-fluid" of Bohm, Vigier *et al.* [10–11]. In sec. 4 we review shortly the work of Schiller [12] on the classical case of the rigid sphere in the electromagnetic field, obtaining in this way the equations describing a vortical fluid. In sec. 5 we show that the limit of the Lagrangian of the Pauli equation for arbitrary spin is the classical Lagrangian of a vortical fluid.

2. QUANTIZATION OF A MECHANICAL SYSTEM

Let us consider an N-dimensional Riemannian space with a metric tensor a_{ik}, and the wave equation for the steady case

$$\nabla^2 \psi + n^2 k_0^2 \psi = 0 \tag{2.1}$$

where n is the refraction index and k_0 the wave number in empty space. We thus have the I scheme (fig. 1). Putting

$$\psi(x) = \mathrm{A}(x)\, e^{ik_0 \mathscr{S}(x)}, \tag{2.2}$$

with $\mathrm{A}(x)$ and $\mathscr{S}(x)$ real functions in (2.1) we obtain

$$\nabla \cdot (\mathrm{A}^2 \nabla \mathscr{S}) = 0 \tag{2.3a}$$

$$(\nabla \mathscr{S})^2 = n^2 + \frac{1}{k_0^2} \frac{\nabla^2 \mathrm{A}}{\mathrm{A}} \,. \tag{2.3b}$$

Eq. (2.3a) is described by the II scheme (fig. 1).

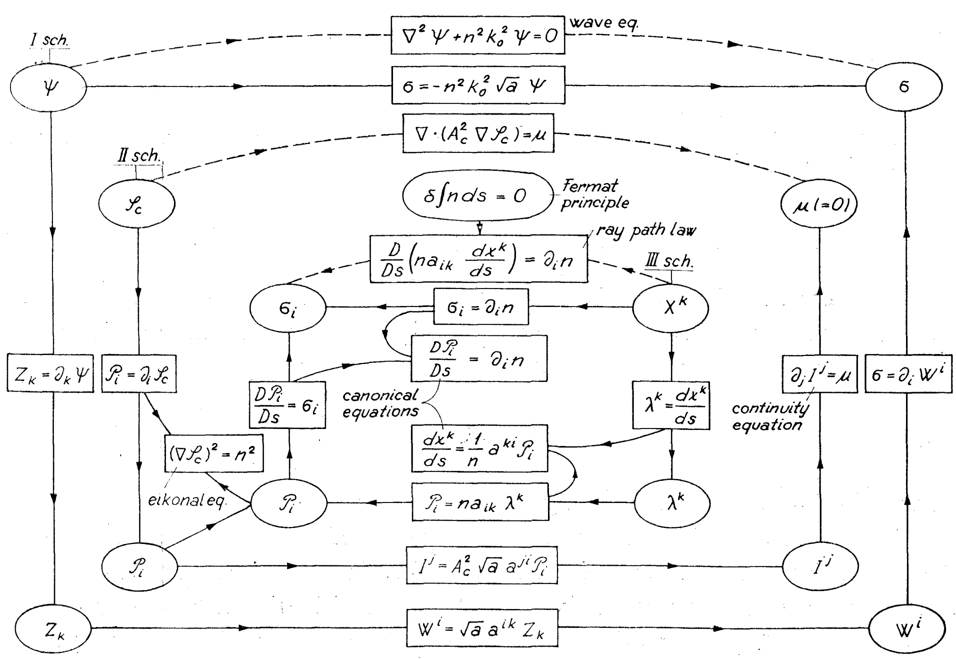

Fig. 1.

The limit process from undulatory to geometrical optics implied to assume in (2.3 b) the amplitude relative variations as negligeable on intervals comparable with the wave length. In this hypothesis (2.3 a) becomes

$$(\nabla \mathscr{S}_c)^2 = n^2 \tag{2.4}$$

where $\mathscr{S}_c$ is the approximate solution. Eq. (2.4) is the eikonal equation and describes the wave fronts of (2.1). Remark that adding to (2.1) a linear term in $\partial_k \psi$ we obtain, by the limit process, the same eikonal eq. (2.4): in this case the transition from geometrical to undulatory optics would exhibit some ambiguities if one considers only the eikonal equation. In other words merely geometrical considerations are not sufficient to determine unambigously the second order propagation equation if one knows only the equation of the wave fronts. If on the contrary to the eikonal equation we add (2.3 a) that involves also the amplitude A, in other words if besides the propagation geometry we consider also the energetic fact, the form of the second order equation results unambigously determined. Till now we have inserted in the scheme only the wave equation and one of the real equivalent equations. The other real equation, i.e. (2.4), in this form is not directly included. But the integration of (2.4) as it is well known, is equivalent, thanks to the method of the Cauchy characteristic, to the integration of a system of 2 N first order equations. Using as parameter the line element $ds = (a_{ik}\, dx^i\, dx^k)^{\frac{1}{2}}$ and putting: $\mathscr{P}_k = \partial_k \mathscr{S}_c$ this system is

$$\frac{dx^i}{ds} = \frac{1}{n}\, a^{ik}\, \mathscr{P}_k \tag{2.5a}$$

$$\frac{D\mathscr{P}_k}{Ds} = \partial_k\, n \tag{2.5b}$$

where D/DS is the absolute derivative.

This system is the canonical form (see [13]) of the system of N second order equations

$$\frac{D}{Ds}\left(n a_{ik} \frac{dx^k}{ds}\right) = \partial_i\, n . \tag{2.6}$$

Eq. (2.6) are the bicharacteristic equations of the wave equation and describe the light ray path in the geometrical optics approximation. As it is well known (2.6) comes from the Fermat variational principle

$$\delta \int n\, ds = 0 . \tag{2.7}$$

So we obtain the III scheme (fig. 1).

Let us now consider a conservative dynamical system for which neither the constraints nor the potential energy V involve the time. Let us suppose that there are N degrees of freedom; then the kinetic energy is given by $T = 1/2\, a_{ik}\, \dot{q}^i\, \dot{q}^k$. The motion of the system can be described as the motion of a point with unitary mass in a Riemannian space with a metric tensor a_{ik}.

The dynamical trajectories are the extremals of the integral

$$\int \sqrt{2\,(\mathrm{E}-\mathrm{V})}\,\mathrm{d}s. \tag{2.8}$$

The equivalent differential equations are the Lagrange equations. They may be written as [14, p. 144].

$$\frac{\mathrm{D}}{\mathrm{D}t}\, p_i(q\,,\dot{q}) = -\,\partial_i \mathrm{V} \tag{2.9}$$

where $\mathrm{D}/\mathrm{D}t$ is the absolute derivative. Since $\mathrm{d}s = [2\,(\mathrm{E}-\mathrm{V})]^{\frac{1}{2}}\,\mathrm{d}t$, the Lagrange equations become

$$\frac{\mathrm{D}p_i}{\mathrm{D}s} = \partial_i \sqrt{2\,(\mathrm{E}-\mathrm{V})}\,, \tag{2.10}$$

and from the definition of p_i it results

$$p_i = \sqrt{2\,(\mathrm{E}-\mathrm{V})}\; a_{ik}\,\frac{\mathrm{d}q^k}{\mathrm{d}s}\,. \tag{2.11}$$

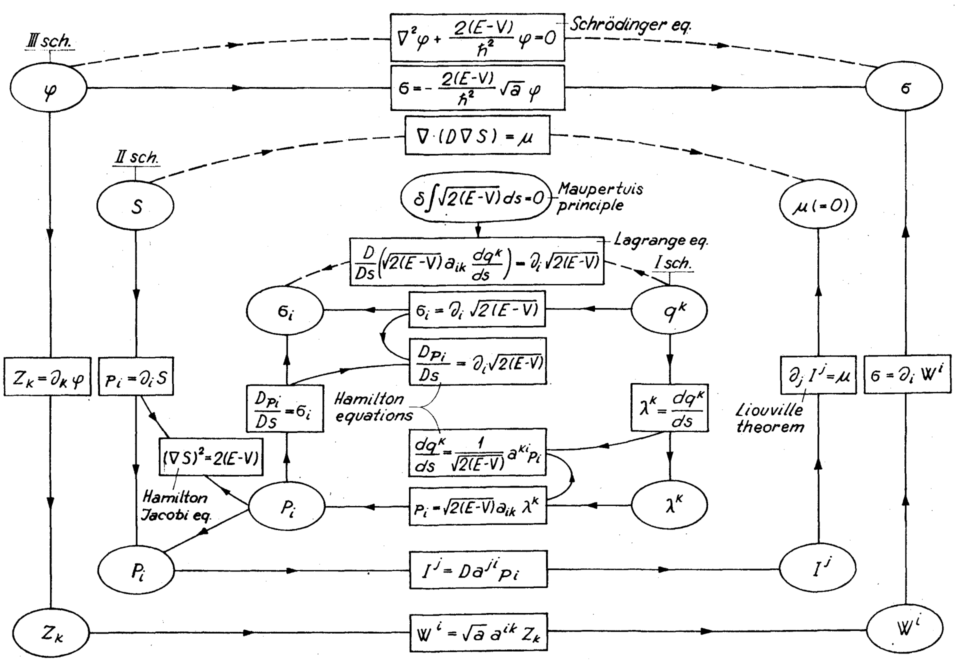

Fig. 2.

The system of 2 N equations (2.10–11) is the canonical form of the Lagrange equations (I sch. fig. 2). They are the Hamilton equations corresponding to the Hamiltonian

$$\mathrm{H} = \tfrac{1}{2}\,a^{ik}\,p_i\,p_k + \mathrm{V}\,. \tag{2.12}$$

The Hamilton equations are the characteristic equations of the HJE

$$(\nabla\mathrm{S})^2 = 2\,(\mathrm{E}-\mathrm{V})\,. \tag{2.13}$$

Usually at this point, remarking the formal analogy between eikonal and HJE, one introduces the Schrödinger equation. But, as we said, there are many second order equations that, in the approximation of geometrical optics, become the eikonal equation. Rot [5, p. 260] has remarked this fact, but he assumes as a postulate that the quantum equation has the form (2.1). On the contrary it is possible to find in CM an equation that is formally analogous to (2.3a) and therefore to deduce rigorously the Schrödinger equation: if $S(q, \alpha)$ is a complete integral of HJE, Van Vleck [6] and Schiller [7] have proved that the scalar density defined in configuration space

$$D(q, \alpha) = \det \left\| \frac{\partial^2 S}{\partial q^i \partial \alpha_k} \right\| \tag{2.14}$$

satisfies the equation

$$\partial_k (D v^k) = 0, \tag{2.15}$$

where

$$v^k = \frac{\partial H}{\partial \left(\dfrac{\partial S}{\partial q^k} \right)}. \tag{2.16}$$

So it is possible to complete the II scheme (fig. 2): if we compare it with the scheme of the steady compressible fluid we notice that S and D play the same role of the velocity potential and fluid density. Therefore D is interpretable as a particle density. If we consider only a particle and we know the constants of motion $\alpha_k' s$ but we do not know the other initial conditions, D may be interpreted as a probability density. (2.15) is substantially equivalent to the Liouville theorem.

If we compare II and III sch. of fig. 1 with II and I of fig. 1, we notice their complete formal analogy. In particular

$$n \to \sqrt{2(E - V)} \tag{2.17 a}$$

$$A_c^2 \sqrt{a} \simeq \psi^* \psi \sqrt{a} \to D. \tag{2.17 b}$$

This analogy imposes that the equation of the WM is

$$\nabla^2 \varphi + \frac{2(E - V)}{\hbar^2} \varphi = 0, \tag{2.18}$$

$\hbar$ being the quantum analogous of $1/k_0$. One cannot say anything about this quantity a priori, every information being deferred to experimental facts: but this is the only thing left to the experiment.

It is possible to generalize the described method and to find the wave equation of a system whose Lagrangian is

$$L = \frac{1}{2} a_{ik}(q, t) \dot{q}^i \dot{q}^k + b_i(q, t) \dot{q}^i + c(q, t) \tag{2.19}$$

and corresponding Hamiltonian

$$H = \frac{1}{2} a^{ik} (p_i - b_i)(p_k - b_k) - c. \tag{2.20}$$

We can apply the described method if we find another Lagrangian and another Hamiltonian which are equivalent to (2.19–20), quadratic and homogeneous in the $\dot{q}$'s and p's and explicitly dependent only on the coordinates. This last requirement may be satisfied considering the time $t = q^0$ as a coordinate and introducing a parameter τ. With regard to the other requirements we remember that, given a Lagrangian that is quadratic and homogeneous in $(N+1)$ velocities with a cyclic coordinate, this coordinate may be eliminated obtaing a Lagrangian that includes linear terms in $\dot{q}$'s. We can therefore consider the Lagrangian (2.19) derived from the Lagrangian: $\mathscr{L} = 1/2\, g_{AB}\, \dot{q}^A\, \dot{q}^B$ $(A, B = 0, 1 \cdots N+1)$, through the elimination of a cyclic coordinate q^{N+1}. The corresponding Hamiltonian is $\mathscr{H} = 1/2\, g^{AB}\, p_A\, p_B$. The tensor g_{AB} is to be determined as a function of a_{ik}, b_i, c. This may be made thanks to a theorem of Eisenhart [15]: the solutions of (2.19–20) are the projection on the hyperplane $(q^0 \cdots q^N)$ of the geodesics with zero length of a Riemannian space with a non definite metric

$$g_{AB} = \begin{pmatrix} 2c & b_k & 1 \\ b_i & a_{ik} & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad ; \quad g^{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & a^{ik} & -b^i \\ 1 & -b^k & b_h\, b^h - 2c \end{pmatrix}. \tag{2.21}$$

Note that $|g| = a$. The Eisenhart theorem may be easily checked considering the $(2N+4)$ Hamilton equations from $\mathscr{H}$. As to the indices 0 and $(N+1)$ they are

$$\frac{dt}{d\tau} = p_{N+1} \qquad \frac{dp_0}{d\tau} = -\frac{1}{2}\, \frac{\partial g^{AB}}{\partial t}\, p_A\, p_B \tag{2.22}$$

$$\frac{dq^{N+1}}{d\tau} = g^{N+1,B}\, p_B \qquad \frac{dp^{N+1}}{d\tau} = 0\,. \tag{2.23}$$

The latter of (2.23) states that p_{N+1} is constant, in accordance with the fact that q^{N+1} is cyclic. If we put this constant equal to 1 (therefore $dt = d\tau$), we obtain what the theorem states. In fact: the other two equations become

$$\frac{dp_0}{dt} = -\frac{\partial H}{\partial t} \qquad \frac{dq^{N+1}}{dt} = -L\,, \tag{2.24}$$

and therefore p_0 coincides with H and q^{N+1} with the action except in sign; the other $2N$ equations coincide with those derived from (2.20). Put: $ds^2 = g_{AB}\, dq^A\, dq^B$; this line element is null for the solutions and therefore $\mathscr{L} = \mathscr{H} = 0$. $\mathscr{H}$ being constant and the terms corresponding to the potential energy being not present, the principle of least action is

$$\delta \int ds = 0\,, \tag{2.25}$$

that is equivalent, as one sees through simple computations, to the Hamilton principle

$$\delta \int L\, dt = 0\,. \tag{2.26}$$

The quantum equation, the numerical value of $\mathscr{H}$ being zero and the terms corresponding to the potential energy being not present, results

$$\nabla^2 \Phi (q^0 \cdots q^{N+1}) = 0 , \tag{2.27}$$

where the Laplacian is expressed in the metric (2.21). Since q^{N+1} is cyclic we may put

$$\Phi = \varphi (t , q^1 \cdots q^N) \, e^{(i/\hbar) q^{N+1}}, \tag{2.28}$$

so obtaining the equation already found by Rot [5, p. 262]. This equation may be simplified putting

$$\psi = \varphi a^{1/4}, \tag{2.29}$$

from which we obtain the wave equation

$$\left[\frac{1}{2} a^{-1/4} \left(\frac{\hbar}{i} \partial_h - b_h\right) a^{1/2} a^{hk} \left(\frac{\hbar}{i} \partial_k - b_k\right) a^{-1/4} - c\right] \psi + \frac{\hbar}{i} \frac{\partial \psi}{\partial t} = 0 . \tag{2.30}$$

This may be obtained writing the Hamiltonian in the form

$$H = \tfrac{1}{2} a^{-1/4} (p_h - b_h) \, a^{1/2} a^{hk} (p_k - b_k) \, a^{-1/4} - c \tag{2.31}$$

and making the substitution

$$p_\mu \to \frac{\hbar}{i} \partial_\mu \qquad (\mu = 0 , 1 , \cdots , N) \tag{2.32}$$

in the equation: $H (q , p , t) + p_0 = 0$. By this way we obtain the generalization of the known Podolsky rule [16].

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