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Wallman Compactifications by Collections of 0—1 Measures

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Topologia. — *Wallman Compactifications by Collections of* 0—1 *Measures*. Nota di MICHAEL J. D'AMBROSA, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Si prova che ogni compattificazione di Wallman è equivalente a una compattificazione generata da una collezione di certe 0—1 misure. Si estende poi il cosiddetto « Portmanteau theorem » di Varadarajan.

In Varadarajan [11] there is a measure-theoretic treatment of the Stone-Cěch compactification βX of a Tychonoff space X. We extend this to a more general framework by showing that every Wallman compactification (as in Frink [4]) is equivalent to a compactification generated by a collection of certain 0—1 measures. We thereby extend the so-called " Portmanteau Theorem " of Varadarajan (Part II, Theorem 2) and Aleksandrov [1] and we complement the work of Frink by giving an explicit integral representation of certain extended functions.

Since we are concerned with Hausdorff compactifications of X, we'll assume throughout that X is a Tychonoff topological space. If A is a subset of X, the complement of A will be denoted by A'; other complements will be denoted by "—".

Note that we establish a one-to-one correspondence between Λ-ultrafilters (where Λ is a normal base, as in Frink) and 0—1 measures. Thus the usual ultrafilter statements have corresponding measure statements. We will only point out those which we need, but the reader will certainly recognize others. For example we can characterize compactness by the fact that all the measures are " fixed ".

§ 1. THE WALLMAN COMPACTIFICATION $\omega(\Lambda, X)$

We begin by reviewing some terminology and results in [4].

1.1. DEFINITION. A collection Λ of closed subsets of X is called a *normal base* for X iff:

(1) Λ is a base for the closed sets of X.

(2) Λ is closed under finite unions and interesections.

(3) Λ is *disjunctive*; i.e. if $x \notin F$ for some closed subset F of X, then there exists some $A \in \Lambda$ such that $x \in A$ and $A \cap F = \varnothing$.

(*) Nella seduta del 16 aprile 1977.

(4) If A and B are disjoint members of Λ, then there exist E and F in Λ such that $A \subset E'$, $B \subset F'$, and $E' \cap F' = \varnothing$.

If, in addition, Λ is closed under countable intersections, it is called a δ-*normal base*.

The following is easy to prove:

1.2. THEOREM. *If* Λ *is a normal base for* X, *then* Λ *is a complete neighborhood system.*

Now let $\omega(\Lambda, X)$ be the collection of all Λ-ultrafilters on X and let $L^* = \{\Phi \in \omega(\Lambda, X) : L \in \Phi\}$ for each $L \in \Lambda$. Let $h(x) = \Phi_x = \{L \in \Lambda : x \in L\}$, Thus $h(x)$ is the unique Λ-ultrafilter which converges to x. Frink has proven that $(\omega(\Lambda, X), h)$ is a Hausdorff compactification of X-generally called a Wallman compactification since Frink generalizes a procedure used in [12]. The question of which compactifications are equivalent to some Wallman compactification has been investigated in [2], [7], and elsewhere.

The following theorem, similar to one in [3], gives us a sufficient condition for a compactification to be Wallman.

1.3 THEOREM. *Let* $(X^\sim, k)$ *be any Hausdorff compactification of* X *and let* Λ *be some normal base for* X. *For each* $L \in \Lambda$, *let* $L^\sim$ *be the closure of* $k(L)$ *in* $X^\sim$. *Suppose*:

(*a*) $(A \cap B)^\sim = A^\sim \cap B^\sim$ *for each* A *and* B *in* Λ.

(*b*) $\Lambda^\sim = \{L^\sim : L \in \Lambda\}$ *is a base for the closed sets of* X.

Then: (1) $\Lambda^\sim$ *is a normal base for* $X^\sim$ (*hence also a complete neighborhood system*). *Furthermore let* $G(z) = \Phi_z = \{L \in \Lambda : z \in L^\sim\}$ *for each* $z \in X^\sim$. *Then we also have* (2) $\Phi_z \in \omega(\Lambda, X)$. (3) $G(L^\sim) = L^* = \{\Phi \in \omega(\Lambda, X) : L \in \Phi\}$. (4) G *is a homeomorphism from* $X^\sim$ *onto* $\omega(\Lambda, X)$ *and* $G \circ k = h$, *so that* $(X^\sim, k)$ *is equivalent to* $(\omega(\Lambda, X), h)$.

Proof. (i) (1), (2), and (4) of 1.1 are easily proven. So assume $z \notin K$, where K is closed in $X^\sim$. By (b) there exists $L \in \Lambda$ such that $K \subset L^\sim$ and $z \notin L^\sim$. For each $y \in L^\sim$ we can find some A_y in Λ such that $z \in A_y^\sim$ and $y \notin A_y^\sim$. By the compactness of $L^\sim$ we can find $A_1, \cdots, A_n$ in Λ such that $z \in A_j^\sim$ for $j = 1, 2, \cdots, n$ and the collection $X^\sim - A_1^\sim, \cdots, X^\sim - A_n^\sim$ covers $L^\sim$. Let $A = \bigcap_{j=1}^{n} A_j^\sim$ to prove (3) of 1.1. So (1) is proven. (ii) It is easy to show that Φ_z is a Λ-filter. Now suppose $L \in \Lambda$ and $L \cap A \neq \varnothing$ for each $A \in \Phi_z$. Suppose $z \notin L^\sim$. By (1) there exists $A \in \Lambda$ such that $z \in A^\sim$ and $A^\sim \cap L^\sim = \varnothing$. Hence $A \in \Phi_z$ and $A \cap L = \varnothing$, which is a contradiction. Thus $z \in L^\sim$, so that $L \in \Phi_z$, proving (2). (iii) Next assume $\Phi_z = \Phi_y$. If $z \neq y$ there exist A and B in Λ such that $y \in A^\sim$, $z \in B^\sim$, and $A^\sim \cap B^\sim = \varnothing$. Thus $A \in \Phi_y$ and $B \in \Phi_z$, which is a contradiction. Hence $z = y$ and G is one-to-one Now let $z \in X^\sim$ and let $L \in \Lambda$. Then $\Phi_z \in G(L^\sim)$ iff $z \in L^\sim$ iff $L \in \Phi_z$ iff $\Phi_z \in L^*$, proving (3). (iv) Clearly G is onto. Let $\Lambda^* = \{L^* : L \in \Lambda\}$. Since $\Lambda^\sim$ and Λ^* are bases for the closed sets of $X^{\sim}$ and $\omega(\Lambda, X)$, G is a homeomorphism. So to prove (4) it remains to show that $G \circ k = h$. So let $x \in X$ and let $k(x) = z \in X^{\sim}$. Now $L \in \Phi_x$ implies $x \in L$; thus $z = k(x) \in k(L) \subset L^{\sim}$, so that $L \in \Phi_z$. So $\Phi_x \subset \Phi_z$. But Φ_x is maximal, so $\Phi_x = \Phi_z$; i.e. $h(x) = G(z) = G(k(x))$.

§ 2. Λ-REGULAR 0—1 MEASURES

Throughout this section Λ will denote a fixed normal base for X. Let $A(\Lambda)$ be the algebra generated by Λ.

2.1. DEFINITION. $M(\Lambda, X)$ is the collection of all set functions μ defined on $A(\Lambda)$ such that:

(*a*) $\mu(\varnothing) = 0$ and $\mu(X) = 1$.

(*b*) $\mu(E) = 0$ or 1 for each E in $A(\Lambda)$.

(*c*) μ is finitely-additive.

(*d*) μ is Λ-*regular*; i.e. $\mu(E) = \sup\{\mu(L) : L \in \Lambda \text{ and } L \subset E\}$ for each E in $A(\Lambda)$.

Any set function satisfying (*a*)–(*c*) is called a "0—1 measure". In addition to the usual properties we have:

2.2. LEMMA. *Let* $\mu \in M(\Lambda, X)$ *and let* E *and* F *be in* $A(\Lambda)$. *Then*:

(1) $\mu(E) = 1$ *iff* $\mu(E') = 0$.

(2) $\mu(E \cup F) = 1$ *iff* $\mu(E) = 1$ *or* $\mu(F) = 1$.

(3) $\mu(E \cap F) = 1$ *iff* $\mu(E) = 1$ *and* $\mu(F) = 1$.

This easily proven lemma implies the following:

2.3. THEOREM. *For each* $L \in \Lambda$, *let* $L^{\wedge} = \{\mu \in M(\Lambda, X) : \mu(L) = 1\}$. *Then*: (1) *For each* E *and* F *in* Λ, $(E \cup F)^{\wedge} = E^{\wedge} \cup F^{\wedge}$ *and* $(E \cap F)^{\wedge} = E^{\wedge} \cap F^{\wedge}$. (2) $\Lambda^{\wedge} = \{L^{\wedge} : L \in \Lambda\}$ *is a base for the closed sets for some topology on* X.

Hereafter $L^{\wedge}$ and $\Lambda^{\wedge}$ will be as above and $M(\Lambda, X)$ will have the topology generated by $\Lambda^{\wedge}$. Now we establish a connection between Λ-ultrafilters and 0—1 measures.

2.4. LEMMA. *Let* Φ *be a* Λ-*ultrafilter. Let* $A(\Phi) = \{E \subset X : L \subset E$ *or* $L \subset E'$ *for some* $L \in \Phi\}$. *Then* $A(\Phi)$ *is an algebra and* $A(\Phi)$ *contains* $A(\Lambda)$.

Proof. (1) Clearly $E \in A(\Phi)$ implies $E' \in A(\Phi)$. Now suppose E and F are in $A(\Phi)$. *Case* (*i*): $K \subset E'$ and $L \subset F'$, where K and L are in Φ. Then $K \cap L \subset E' \cap F' = (E \cup F)'$. Since $K \cap L \in \Phi$, $E \cup F \in A(\Phi)$. *Case* (*ii*): $L \subset E$ or $L \subset F$ for some $L \in \Phi$. Then $L \subset E \cup F$, so that $E \cup F \in A(\Phi)$. Therefore $A(\Phi)$ is an algebra. (2) Now let $L \in \Lambda$. If $L \subset F'$ for some $F \in \Phi$, then $F \subset L'$ and so $L \in A(\Phi)$. If $L \not\subset F'$ for each $F \in \Phi$, then $L \cap F \neq \varnothing$ for each $F \in \Phi$; thus $L \in \Phi \subset A(\Phi)$. Therefore $\Lambda \subset A(\Phi)$, so that $A(\Lambda) \subset A(\Phi)$.

2.5. THEOREM. *Let* Φ *be a* Λ*-ultrafilter. For each* $E \in A(\Lambda)$ *let* $\mu_\Phi(E) = 1$ *if* $L \subset E$ *for some* $L \in \Phi$ *and* $\mu_\Phi(E) = 0$ *if* $L \subset E'$ *for some* $L \in \Phi$. *Then* $\mu_\Phi \in M(\Lambda, X)$.

Proof. By 2.4 one of the above conditions must be true. Since both can't be true μ_Φ is well-defined and clearly satisfies (*a*) and (*b*) of 2.1. Now suppose E and F are disjoint sets in $A(\Lambda)$. Assume $L \subset E$ for some $L \in \Phi$. Then $L \subset E \cup F$ and $L \subset F'$. Thus $\mu_\Phi(E \cup F) = 1 = 1 + 0 = \mu_\Phi(E) + \mu_\Phi(F)$. Similar results hold if $L \subset F$ for some $L \in \Phi$. So now assume $L \not\subset E$ and $L \not\subset F$ for each $L \in \Phi$. By 2.4 there exist A and B in Φ such that $A \subset E'$ and $B \subset F'$, so that $A \cap B \subset (E \cup F)'$. Hence $\mu_\Phi(E \cup F) = 0 = 0 + 0 = \mu_\Phi(E) + \mu_\Phi(F)$. Thus 2.1 (*c*) is proven and (*d*) follows from 2.4.

Now let $\Phi = \Phi_x$ for some (fixed) $x \in X$ and denote μ_Φ by μ_x. Applying 2.4 and 2.5 we easily establish the following for these "fixed" 0—1 measures.

2.6. COROLLARY. *Let* x *be a fixed element of* X *and, for each* $E \in A(\Lambda)$, *define* $\mu_x(E) = 1$ *iff* $x \in L \subset E$ *for some* $L \in \Lambda$ *and* $\mu_x(E) = 0$ *iff* $x \in L \subset E'$ *for some* $L \in \Lambda$. *Then* $\mu_x \in M(\Lambda, X)$. *Furthermore* $\mu_x(E) = 1$ *iff* $x \in E$ *and* $\mu_x(E) = 0$ *iff* $x \notin E$.

Let $V(L) = M(\Lambda, X) - L^\wedge$ for each $L \in \Lambda$. Thus $\{V(L) : L \in \Lambda\}$ is a base for the open sets of $M(\Lambda, X)$. Note $\mu \in V(L)$ iff $\mu \notin L^\wedge$ iff $\mu(L) \neq 1$ iff $\mu(L) = 0$. So $V(L) = \{\mu \in M(\Lambda, X) : \mu(L) = 0\}$.

2.7. LEMMA. *For each* $x \in X$ *let* $g(x) = \mu_x$. *Then*: (1) g *is one-to-one.* (2) $g(L') = V(L) \cap g(X)$ *for each* $L \in \Lambda$. (3) $g^{-1}(V(L)) = L'$ *for each* $L \in \Lambda$.

Proof. (1) Assume $x \neq y$. Then there exists $L \in \Lambda$ such that $x \in L$, $y \notin L$; thus $\mu_x(L) = 1$ and $\mu_y(L) = 0$. So $\mu_x \neq \mu_y$ or $g(x) \neq g(y)$. (2) Let $x \in X$ and let $g(x) = \mu_x$. Since g is one-to-one $\mu_x \in g(L')$ iff $x \in L'$ iff $\mu_x(L) = 0$ iff $\mu_x \in V(L) \cap g(X)$. (3) $L' = g^{-1}(g(L')) = g^{-1}(V(L) \cap g(X)) = g^{-1}(V(L)) \cap \cap g^{-1}(g(X)) = g^{-1}(V(L)) \cap X = g^{-1}(V(L))$.

2.8. THEOREM. $(M(\Lambda, X), g)$ *is a Hausdorff compactification of* X.

Proof. (i) 2.7 (2) implies g is open; 2.7 (3) implies g is continuous. Thus g is homeomorphism. Now suppose $V(L) \neq \varnothing$ for some $L \in \Lambda$. Then $L \neq X$, so that $L' \neq \varnothing$ and thus $g(L') \neq \varnothing$. So by 2.7 (2) $V(L) \cap g(X) \neq \varnothing$. Therefore $g(X)$ isdense in $M(\Lambda, X)$. (ii) Suppose $\mu \neq \nu$ in $M(\Lambda, X)$. Then there exists $A \in \Lambda$ such that $\mu(A) = 1$ and $\nu(A) = 0$, or conversely (if $\mu = \nu$ on Λ, then $\mu = \nu$ on $A(\Lambda)$ by regularity). Since $\nu(A') = 1$, there exists $B \subset A'$ such that $B \in \Lambda$ and $\nu(B) = 1$. Now $A \cap B = \varnothing$, so there exist E and F in Λ such that $A \subset E'$, $B \subset F'$, and $E' \cap F' = \varnothing$. Note $A \subset E'$ implies $\mu(E') = 1$; hence $\mu(E) = 0$ or $\mu \in V(E)$. Similarly $\nu \in V(F)$. Finally $V(E) \cap V(F) = M(\Lambda, X) - (E^\wedge \cup F^\wedge) = M(\Lambda, X) - (E \cup F)^\wedge = M(\Lambda, X) - M(\Lambda, X) = \varnothing$. Thus $M(\Lambda, X)$ is a Hausdorff space. (iii) Let $\Omega^\wedge = \{L_\alpha^\wedge\}$ be any collection of basic closed sets (i.e. $L_\alpha \in \Lambda$) having the finite intersection property. To prove $M(\Lambda, X)$ is compact it suffices to show $\Omega^\wedge$ has a non-empty intersection. Let $\Omega = \{L_\alpha\}$, so that Ω also has the finite intersection property (2.3). So Ω can be extended to a Λ-ultrafilter Φ. Let $\mu = \mu_\Phi$, as in 2.5. Now $\mu(L) = 1$ for each $L \in \Phi$, so that $\mu \in L^\wedge$ for each $L \in \Phi$. Thus $\mu \in L_\alpha^\wedge$ for each α, proving $M(\Lambda, X)$ is compact, which completes the proof of 2.8.

2.9. LEMMA. *For each* $L \in \Lambda$, $\overline{g(L)} = L^\wedge$ (*where the closure is taken in* $M(\Lambda, X)$).

Proof. $x \in L$ implies $g(x) = \mu_x \in L^\wedge$, since $\mu_x(L) = 1$. Thus $g(L) \subset L^\wedge$. Since $L^\wedge$ is closed $\overline{g(L)} \subset L^\wedge$. Now let $\mu \in L^\wedge$ and let U be any neighborhood of μ. Then there exists $A \in \Lambda$ such that $\mu \in V(A) \subset U$. Now $\mu(A) = 0$ and $\mu(L) = 1$, so that $L \not\subset A$. So choose any $x \in L - A$; then $g(x) = \mu_x \in g(L)$. But $\mu_x(A) = 0$, so that $\mu_x \in V(A) \subset U$. Hence $U \cap g(L) \neq \varnothing$, so that $\mu \in \overline{g(L)}$; i.e. $L^\wedge \subset \overline{g(L)}$.

2.10 THEOREM. *Let* $\omega(\Lambda, X)$ *and* h *be as in* § 1. *Define* G *from* $M(\Lambda, X)$ *to* $\omega(\Lambda, X)$ *by* $G(\mu) = \Phi_\mu = \{L \in \Lambda : \mu(L) = 1\}$. *Then* G *is a homeomorphism from* $M(\Lambda, X)$ *onto* $\omega(\Lambda, X)$ *and* $G \circ g = h$; i.e. *the compactifications* $(\omega(\Lambda, X), h)$ *and* $(M(\Lambda, X), g)$ *are equivalent.*

The proof of this theorem follows from the above and 1.3. Furthermore we can easily show that $G^{-1}(\Phi) = \mu_\Phi$ (as in 2.5) for each $\Phi \in \omega(\Lambda, X)$. We close this section with a result from [4].

2.11 DEFINITION. A function f from X into R (reals) is said to be Λ-*uniformly continuous* iff for each $\delta > 0$ there exist $L_1, \cdots, L_n$ in Λ such that $X = \bigcup_{s=1}^{n} L_s'$ and if $x, y \in L_s'$ (for some $s = 1, \cdots, n$), then $|f(x) - f(y)| < \delta$.

2.12. THEOREM. *Let* $f \in C(X)$. *Then* f *has a continuous extension* F *to* $\omega(\Lambda, X)$ (i.e. $F \circ h = f$) *iff* f *is* Λ-*uniformly continuous.*

Note that Frink's theorem clearly applies to any compactification equivalent to $\omega(\Lambda, X)$-in particular it applies to $M(\Lambda, X)$.

§ 3. INTEGRATION AND $M(\Lambda, X)$

Since the elements of $M(\Lambda, X)$ are measures, it is only natural to define an integral. However, we mut be careful since the measures are only finitely-additive. Thus we define a "measurable function" as follows (where Λ is a normal base):

3.1. DEFINITION. A function f from X into R is Λ-*measurable* iff $f^{-1}(I) \in A(\Lambda)$ for each (finite or infinite) interval I of R.

Thus if f is bounded and Λ-measurable we can use the classical Lebesgue approach to define $\int_E f \, d\mu$ for any $\mu \in M(\Lambda, X)$ and any $E \in A(\Lambda)$. See, for example, [10], pp. 332–333. We will denote $\int f \, d\mu$ simply by $\int f \, d\mu$. The integral will have the usual (finitely-additive) properties ([10], pp. 334–337).

Unfortunately a continuous function is not necessarily Λ-measurable, so we need a condition stronger than continuity (in this connection see [1]).

3.2. DEFINITION. Let $f: X \to R$. Then f is *Λ-continuous* iff $f^{-1}(K) \in \Lambda$ for each closed set K in R.

Note that Λ-continuous implies Λ-measurable (and continuous). The following is easily proven:

3.3. LEMMA *If Λ is a δ-normal base, then the following are equivalent:* (1) *f is Λ-continuous.* (2) $\{x \in X : f(x) > \alpha\}$ *and* $\{x \in X : f(x) < \alpha\}$ *are complements of sets in Λ for each α in* R. (3) $\{x \in X : f(x) \geq \alpha\}$ *and* $\{x \in X : f(x) \leq \alpha\}$ *are in Λ for each α in* R.

The following version of Urysohn's Lemma is proved in [1] (1940, p. 317, Lemma 2).

3.4. LEMMA. *Let Λ be a δ-normal base and let* L *and* M *be any disjoint sets of Λ. Then there exists a Λ-continuous function f such that* $0 \leq f \leq 1$, $f(L) = \{0\}$, *and* $f(M) = \{1\}$.

3.5. LEMMA. *Let Λ be a δ-normal base for* X, *let* $\mu \in M(\Lambda, X)$, *and let* $\{\mu_\alpha\}$ *be a net in* $M(\Lambda, X)$. *Suppose* $\int f \, d\mu_\alpha \to \int f \, d\mu$ *for each bounded Λ-continuous function f. Then* $\overline{\lim}\, \mu_\alpha(L) \leq \mu(L)$ *for each* $L \in \Lambda$ (*so that* $\underline{\lim}\, \mu_\alpha(L') \geq \mu(L')$ *also*).

Proof. If $\mu(L) = 1$ the result is clear, so assume $\mu(L) = 0$. Hence $\mu(L') = 1$, so that there exists $M \in \Lambda$ such that $M \subset L'$ and $\mu(M) = 1$. Note $\mu(M') = 0$ and $M \cap L = \varnothing$, so by 3.4, there exists a Λ-continuous function f such that $0 \leq f \leq 1$, $f(M) = \{0\}$, and $f(L) = \{1\}$. Thus $\int f \, d\mu = \int_M f \, d\mu + \int_{M'} f \, d\mu = 0 + 0 = 0$. Suppose $0 < \delta < 1$. Then there exists β such that $\left| \int f \, d\mu - \int f \, d\mu_\alpha \right| < \delta$ for each $\alpha > \beta$. Thus $0 \leq \int f \, d\mu_\alpha < \delta$ for each $\alpha > \beta$. If $\alpha > \beta$, then $\delta > \int f \, d\mu_\alpha \geq \int f \, d\mu_\alpha = \mu_\alpha(L)$. So $\mu_\alpha(L) = 0$ for each $\alpha > \beta$, from which lemma follows.

3.6. LEMMA. *Let Λ be a normal base for* X, *and let f be bounded and Λ-continuous. Let* $F(t) = \{x \in X : f(x) \leq t\}$. *For some (fixed)* $\mu \in M(\Lambda, X)$ *define* $\phi(t) = \mu(F(t))$. *Assume* $-N < -K \leq f(x) \leq K < N$ $(K \geq 0, N > 0)$ *for each* $x \in X$. *Then:* (*a*) *there exists some real number z such that* $\phi(t) = 1$ *for* $t > z$ *and* $\phi(t) = 0$ *for* $t < z$, *and* (*b*) $\int f \, d\mu = z = N - \int_{-N}^{N} \phi(t) \, dt$.

Proof. Note that ϕ is increasing, $\phi(t) = 0$ or 1 for each t, $\phi(N) = 1$ and $\phi(-N) = 0$, proving (*a*). Now let $\delta > 0$ be given. We may assume

$-N < z - \delta < z + \delta < N$, so that $\varnothing = F(-N) \subset F(z-\delta) \subset F(z+\delta) \subset$ $\subset F(N) = X$. If we let $A = F(N) - F(z+\delta)$, $B = F(z+\delta) - F(z-\delta)$, and $C = F(z-\delta)$, then A, B, and C are disjoint sets whose union is X. Note $\mu(A) = 0$ and $\mu(C) = 0$. Thus $\int f \, d\mu = \int_B f \, d\mu$, and $\mu(B) = 1$. But $z - \delta < f(x) \leq z + \delta$ for each $x \in B$, so that $z - \delta \leq \int f \, d\mu \leq z + \delta$. Thus $\int f \, d\mu = z$ and the rest of (*b*) follows by integrating.

The following is the converse of 3.5. Note that we need not assume Λ is δ-normal.

3.7 LEMMA. *Let* Λ *be any normal base for* X; *let* $\mu \in M(\Lambda, X)$ *and let* $\{\mu_\alpha\}$ *be a net in* $M(\Lambda, X)$. *Suppose for each* $L \in \Lambda$, $\overline{\lim}\, \mu_\alpha(L) \leq \mu(L)$ (*so that* $\underline{\lim}\, \mu_\alpha(L') \geq \mu(L')$ *also*). *Then* $\int f \, d\mu_\alpha \to \int f \, d\mu$ *for each bounded* Λ-*continuous function f*.

Proof. Assume $-N < -K \leq f \leq K < N$. Let $F_t = \{x \in X : f(x) \leq t\}$ and let $G_t = \{x \in X : f(x) < t\}$, so that F_t and G_t' are in Λ for each t. Let $\phi(t) = \mu(F_t)$ and $\phi_\alpha(t) = \mu_\alpha(F_t)$. Thus $\underline{(1)}$ $\overline{\lim}\, \phi_\alpha(t) \leq \phi(t)$ for each t. Note $s < t$ implies $F_s \subset G_t \subset F_t$, so that $\mu(F_s) \leq \mu(G_t) \leq \mu(F_t)$ and $\mu_\alpha(F_s) \leq \mu_\alpha(G_t) \leq \mu_\alpha(F_t)$. So, for each $\delta > 0$, $\mu(G_t) \geq \mu(F_{t-\delta}) = \phi(t-\delta)$. Also $\mu_\alpha(G_t) \leq$ $\leq \mu_\alpha(F_t) = \phi_\alpha(t)$. Thus, for each $\delta > 0$; , $\phi(t-\delta) \leq \mu(G_t) \leq \underline{\lim}\, \mu_\alpha(G_t') \leq \underline{\lim}\, \mu_\alpha(F_t) = \underline{\lim}\, \phi_\alpha(t)$. So we've shown: $\underline{(2)}$ $\phi(t-\delta) \leq \underline{\lim}\, \phi_\alpha(t)$ for each t and each $\delta > 0$. Since $\int f \, d\mu = N - \int_{-N}^{N} \phi(t) \, dt$ and $\int f \, d\mu_\alpha = -N \int_{-N}^{N} \phi_\alpha(t) \, dt$, it suffices to show that $\lim \int_{-N}^{N} \phi_\alpha(t) \, dt = \int_{-N}^{N} \phi(t) \, dt$. So let z be as in 3.6 (for μ), and let $\delta > 0$. Now $\phi(z-\delta) = 0$, so that $\overline{\lim}\, \phi_\alpha(z-\delta) \leq 0$ by $\underline{(1)}$; therefore $\overline{\lim}\, \phi_\alpha(z-\delta) = 0$. So there exists β such that $\phi_\alpha(z-\delta) = 0$ for $\alpha > \beta$, so that $\phi_\alpha(t) = 0$ for all $t \leq z - \delta$ and all $\alpha > \beta$. Thus $\int_{-N}^{N} \phi_\alpha(t) \, dt = \int_{z-\delta}^{N} \phi_\alpha(t) \, dt \leq N - (z-\delta) = (N-z) + \delta$ for each $\alpha > \beta$. Hence $\overline{\lim} \int_{-N}^{N} \phi_\alpha(t) \, dt \leq (N-z) + \delta$. Clearly then $\overline{\lim} \int_{-N}^{N} \phi_\alpha(t) \, dt \leq N - z$. Similarly by $\underline{(2)}$ $\phi(t-\delta) \leq \underline{\lim}\, \phi_\alpha(t)$ for each t and each $\delta > 0$. Thus $1 = \phi((z+2\delta)-\delta) \leq \underline{\lim}\, \phi_\alpha(z+2\delta)$. So there exists γ such that $\phi_\alpha(z+2\delta) = 1$ for each $\alpha > \gamma$. Thus $\phi_\alpha(t) = 1$ for each $t \geq z + 2\delta$ and for each $\alpha > \gamma$. So $\int_{-N}^{N} \phi_\alpha(t) \, dt \geq \int_{z+\alpha\delta}^{N} \phi_\alpha(t) \, dt = N - (z + 2\delta) =$

$= (N - z) - 2\delta$ for each $\alpha > \gamma$. Thus $\underline{\lim} \int_{-N}^{N} \phi_\alpha(t)\, dt \geq (N - z) - 2\delta$, impliyng that $\underline{\lim} \int_{-N}^{N} \phi_\alpha(t)\, dt \geq N - z$, from which conclusion follows.

We are now in a position to prove the following extension of the so-called " Portmanteau Theorem ", where convergence of nets of measures is related to convergence of integrals. Our theorem generalizes a theorem in [11] (Part II, Theorem 2).

3.8. THEOREM. *Let* Λ *be a normal base for* X, *let* $\mu \in M(\Lambda, X)$, *and let* $\{\mu_\alpha\}$ *be a net in* $M(\Lambda, X)$. *Consider the following statements*:

(1) $\mu_\alpha \to \mu$ (*i.e. for each basic neighborhood* $V(L)$ *of* μ, *where* $L \in \Lambda$ *there exists* β *such that* $\mu_\alpha \in V(L)$ *whenever* $\alpha > \beta$.

(2) $\overline{\lim}\, \mu_\alpha(L) \leq \mu(L)$ *for each* $L \in \Lambda$.

(3) $\underline{\lim}\, \mu_\alpha(L') \geq \mu(L')$ *for each* $L \in \Lambda$.

(4) $\mu_\alpha(L) \to \mu(L)$ *whenever* $L \in \Lambda$ *and* $\mu(L) = 0$ (i.e. *there exists* β *such that* $\mu_\alpha(L) = 0$ *for* $\alpha > \beta$).

(5) $\int f\, d\mu_\alpha \to \int f\, d\mu$ *for each bounded* Λ*-continuous function* f.

Then: (*a*) (1) *through* (4) *are equivalent.* (*b*) (1) *through* (4) *imply* (5). (*c*) *If* Λ *is a* δ*-normal base, all five are equivalent.*

Proof. Clearly (2) and (3) are equivalent. Now assume (2) is true and $\mu(L) = 0$ for some $L \in \Lambda$. Then $0 \leq \underline{\lim}\, \mu_\alpha(L) \leq \overline{\lim}\, \mu_\alpha(L) \leq \mu(L) = 0$. Hence (2) implies (4). Next assume (4) is true and assume $\mu \in V(L)$ for some $L \in \Lambda$. Thus $\mu(L) = 0$, so that there exists β such that $\mu_\alpha(L) = 0$ for $\alpha > \beta$. Hence $\mu_\alpha \in V(L)$ for $\alpha > \beta$, proving (4) implies (1). Now assume (1) is true. If $\mu(L) = 1$, (2) clearly holds; so assume $\mu(L) = 0$ ($L \in \Lambda$). Thus $\mu \in V(L)$ and there exists β such that $\mu_\alpha \in V(L)$ whenever $\alpha > \beta$; i.e. $\mu_\alpha(L) = 0$ for $\alpha > \beta$. Thus $\overline{\lim}\, \mu_\alpha(L) = 0 = \mu(L)$, proving (1) implies (2). So (*a*) is proven; (*b*) and (*c*) now follow from 3.7 and 3.5.

We close by showing a connection between Λ-continuous and Λ-uniformly continuous functions and by deriving an integral representation for the extension to $M(\Lambda, X)$. This complements a result of Frink [4] already noted and Varadarajan [11].

3.9. THEOREM. *Let* Λ *be a normal base for* X *and assume* f *is bounded and* Λ*-continuous. Then*: (1) f *is* Λ*-uniformly continuous and hence has a continuous extension* F *to* $M(\Lambda, X)$.

(2) $F(\mu) = \int f\, d\mu$ *for each* $\mu \in M(\Lambda, X)$.

Proof. It suffices to define F as in (2) and prove that F is a continuous extension of f to $M(\Lambda, X)$ (which, of course, is unique). So let $\mu \in M(\Lambda, X)$

and let $\{\mu_\alpha\}$ be any net in $M(\Lambda, X)$ such that $\mu_\alpha \rightarrow \mu$. By 3.8 $\int f\,d\mu_\alpha \rightarrow \int f\,d\mu$; i.e. $F(\mu_\alpha) \rightarrow F(\mu)$, proving F is continuous. Now let $g(x) = \mu_x$ as in § 2. We need to show $F \cdot g = f$. So let $F_t = \{y \in X : f(y) \leq t\}$ and let $\phi(t) = \mu_x(F_t)$ as in 3.6 (for fixed x). Note $x \in F_t$ iff $f(x) \leq t$. So $\mu_x(F_t) = 1$ if $t > f(x)$ and $\mu_x(F_t) = 0$ if $t < f(x)$. Thus, as in 3.6, $\int f\,d\mu = z = f(x)$. Thus $F(g(x)) = F(\mu_x) = \int f\,d\mu_x = f(x)$.

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