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The construction of Gysin—Thom homomorphism Between the Cohomologies of Complex Quadric and the Complex Projective space with integer Coefficients

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Geometria algebrica. — *The construction of Gysin–Thom homomorphism Between the Cohomologies of Complex Quadric and the Complex Projective space with integer Coefficients.* Nota di MONIR S. MORSY, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — La costruzione dell'omomorfismo indicato nel titolo viene effettuata poggiando su decomposizioni in celle di una quadrica e di uno spazio proiettivo nel campo complesso.

INTRODUCTION

Let $P_{n+1}(C)$ and $Q_n(C)$ denote, respectively, the complex $(n+1)$-dimensional projective space and the n-dimentional complex quadric. If $(z_0, z_1, \cdots, z_{n+1})$ denote homogeneous complex coordinates on $P_{n+1}(C)$, then $Q_n(C)$ in $P_{n+1}(C)$ is given by the equation

$$\sum_{j=0}^{n+1} z_j^2 = 0 .$$

We propose in this paper to explain the following

CONSTRUCTION. *Given* $Q_n(C)$ *and* $P_{n+1}(C)$, *then we construct the Gysin-Thom homomorphism*

$$f^*: \quad H^{2(n-r)}(Q_n(C); Z) \to H^{2(n+1-r)}(P_{n+1}(C); Z)$$

by using cellular decompositions of $P_{n+1}(C)$ *and* $Q_n(C)$.

The paper is divided into two sections: § 1 is devoted for some preliminaries; the construction takes place in § 2.

The numbers in square brackets refer to the bibliography at the end.

§ 1. PRELIMINARIES [see e.g. 1 and 2]

DEFINITION 1. *Let p and q be two distinct points in* $P_{n+1}(C)$ *having respective coordinates* $(z_0, z_1, \cdots, z_{n+1}), (z_0', z_1', \cdots, z_{n+1}')$. *$p$ and q are said to be* conjugate *with respect to* $Q_n(C)$ *if* $\sum_{j=0}^{n+1} z_j z_j' = 0$.

DEFINITION 2. *The locus of all points conjugate to a given r-complex projective subspace* $P_r(C)$ *of* $P_{n+1}(C)$ *with respect to* $Q_n(C)$ *is an* $(n-r)$*-complex projective subspace* $P_{n-r}(C)$ *which of* $P_{n+1}(C)$ *is called the* polar space *of* $P_r(C)$ *with respect to* $Q_n(C)$.

(*) Nella seduta dell'8 marzo 1975.

DEFINITION 3. *If* $n = 2r$ *or* $n = 2r + 1$, *the quadric* $Q_n(C)$ *contains complex projective spaces of dimension* r *and does not contain any complex projective space of dimension higher than* r. *These are called the generators for* $Q_n(C)$.

We have to note that:

a) If $b = 2r$, the set of generators of $Q_n(C)$ constitutes two distinct families.

b) If $n = 2r + 1$, the set of generators of $Q_n(C)$ constitutes only one continuous family.

c) The intersection of two distinct generators $P_r(C)$ and $P_r'(C)$ of the *same family* is an $(r - 2s)$-complex projective space, where s is an integer such that $r - 2s \geq 1$. But if they belong to *two different families*, then their intersection will be an $(r - 2s - 1)$-complex projective space, where s is an integer such that $r - 2s \geq 0$.

			r–even	*r–odd*
d) $P_r(C) \cap P_r'(C)$	of	the same family	one point in common	disjoint
		different families	disjoint	one point in common.

DEFINITION 4. *Let* $q \in Q_n(C)$ *and let* $P_n(C)$ *be the polar plane of* q *with respect to* $Q_n(C)$. $P_n(C) \cap Q_n(C)$ *is* the quadric cone Σ_{n-1} *having* q *as its vertex*.

Given two distinct points q and $q' \in Q_n(C)$ with respective polar planes $P_n(C)$ and $P_n'(C)$ with respect to $Q_n(C)$. The *central projection* from q onto $P_n'(C)$ puts in $(1 - 1)$-correspondence $P_n' - (P_n(C) \cap P_n'(C))$ and $Q_n(C) - \Sigma_{n-1}$ which is then homeomorphic to an algebraic open cell.

Let $P_k(C)$ $(k < r)$ be a generator of $Q_n(C)$ and $P_{n-k}(C)$ be its polar space with respect to $Q_n(C)$, then $P_k(C)$ is the vertex of a $(k + 1)$-fold degenerate quadric $\Sigma_{n-k-1} = Q_n(C) \cap P_{n-k}(C)$.

Let $q' \in \Sigma_{n-k-1}$ such that $q' \notin P_k(C)$ and let $P_{k+1}(C)$ be the generator of $Q_n(C)$ which contains $P_k(C)$ and q'. The polar space $P_{n-k-1}(C)$ of $P_{k+1}(C)$, with respect to $Q_n(C)$, intersects $Q_n(C)$ in a $(k + 2)$-fold degenerate quadric Σ_{n-k-2}. Every point of $\Sigma_{n-k-1} - \Sigma_{n-k-2}$ corresponds in a unique manner to a straight line in $P_{n-k}(C)$ which passes through q' and does not lie on $P_{n-k-1}(C)$. Thus $\Sigma_{n-k-1} - \Sigma_{n-k-2}$ is homeomorphic to an algebraic open cell.

§ 2. THE CONSTRUCTION

Let us first construct the cellular decomposition of each of $P_{n+1}(C)$ and $Q_n(C)$.

(1) The cellular decomposition of $P_{n+1}(C)$ goes as follows. One can find the following sequence $\{P_r(C)\}_{0 \leq r \leq n+1}$ of complex projective subspaces of $P_{n+1}(C)$ such that:

$$P_{n+1}(C) \supset P_n(C) \supset \cdots \supset P_1(C) \supset P_0(C)\,.$$

The difference of any two successive projective subspaces constitutes an algebraic open cell. That sequence then defines a subdivision of $P_{n+1}(C)$ into algebraic open cells. These cells constitute the bases for the homology group $H_*(P_{n+1}(C);Z)$.

(2) The cellular decomposition of $Q_n(C)$ is constructed as follows.

(i) *n is odd* $(= 2r+1)$:

Let us consider a sequence $\{P_k(C)\}_{0\leq k\leq r}$ of generators of $Q_n(C)$, such that

$$P_r(C)\supset P_{r-1}(C)\supset\cdots\supset P_1(C)\supset P_0(C).$$

Correspondingly we obtain for $j<r$ the sequence of the $(j+1)$-fold degenerate quadrics $\Sigma_{n-j-1}=Q_n(C)\cap P_{n-j}(C)$, where $P_{n-j}(C)$ is the polar space of $P_j(C)$ with respect to $Q_n(C)$. In particular, if $j=r$, then Σ_{n-r-1} reduces to the generator $P_r(C)$. Thus one has a sequence

$$Q_n,\Sigma_{2r},\cdots,\Sigma_{r+1},P_r(C),\cdots,P_1(C),P_0(C),$$

such that

$$Q_n(C)\supset\Sigma_{2r}\supset\cdots\supset\Sigma_{r+1}\supset P_r(C)\supset\cdots\supset P_1(C)\supset P_0(C).$$

The difference of any successive algebraic varieties of that sequence is homeomorphic to an algebraic open cell. That sequence then defines a subdivision of $Q_{2r+1}(C)$ into algebraic open cells. These cells constitute the bases for the homology group $H_*(Q_{2r+1}(C);Z)$. This homology group has no torsion coefficients and $KI(P_j(C)\cdot\Sigma_{n-j})=1$ [1].

(ii) *n is even* $(=2r)$:

Consider in $Q_n(C)$ the following sequence of complex projective spaces

$$P_r(C)\supset P_{r-1}(C)\supset\cdots\supset P_1(C)\supset P_0(C).$$

Every generator $P_j(C)$, of this sequence, defines in the usual manner a $(k+1)$-fold degenerate quadric $\Sigma_{n-j-1}=Q_n(C)\cap P_{n-j}(C)$.

If $j=r-1$, then Σ_{n-j-1} reduces to two generators $P_r(C),P_r'(C)$, of different families, with $P_{r-1}(C)$ as their intersection. The bases of $H_*(Q_{2r}(C);Z)$ in this case are then formed by:

$$Q_{2r}(C),\Sigma_{2r-1},\cdots,\Sigma_{r+1},P_r(C),P_r(C),P_{r-1}(C),\cdots,P_1(C),P_0(C).$$

We also have

$$KI(P_j(C)\cdot\Sigma_{n-j})=1 \qquad \text{if } j\neq r$$

$$KI(P_r(C)\cdot P_r'(C))=\begin{cases}1 & \text{if } r \text{ is odd}\\ 0 & \text{if } r \text{ is even}\end{cases}$$

(1) $KI(x\cdot y)$ denotes the Kronecker index of $x\cdot y$.

and

$$\mathrm{KI}\,(\mathrm{P}_r(\mathrm{C})\cdot\mathrm{P}_r(\mathrm{C})) = \mathrm{KI}\,(\mathrm{P}'_r(\mathrm{C})\cdot\mathrm{P}'_r(\mathrm{C})) = \begin{cases} 0 & \text{if } r \textit{ is odd} \\ 1 & \text{if } r \textit{ is even.} \end{cases}$$

But, since $Q_n(C)$ and $P_{n+1}(C)$ are oriented compact manifolds [2], then by the Poincaré duality lemma [2], there exist isomorphisms

$$*n+1:\quad \mathrm{H}_{2r}(\mathrm{P}_{n+1}(\mathrm{C});\mathrm{Z}) \cong \mathrm{H}^{2(n+1-r)}(\mathrm{P}_{n+1}(\mathrm{C});\mathrm{Z}),$$

and

$$*n:\quad \mathrm{H}_{2r}(\mathrm{Q}_n(\mathrm{C});\mathrm{Z}) \cong \mathrm{H}^{2(n-r)}(\mathrm{Q}_n(\mathrm{C});\mathrm{Z}).$$

Thus we are now in situation to construct f^* from the commutativity of the following diagram

$$\begin{tikzcd} \mathrm{H}^{2(n-r)}(\mathrm{Q}_n(\mathrm{C});\mathrm{Z}) \arrow[r, "f^*"] & \mathrm{H}^{2(n+1-r)}(\mathrm{P}_{n+1}(\mathrm{C});\mathrm{Z}) \\ \mathrm{H}_{2r}(\mathrm{Q}_n(\mathrm{C}),\mathrm{Z}) \arrow[r, "i_*"] \arrow[u, "*_n", "\wr"'] & \mathrm{H}_{2r}(\mathrm{P}_{n+1}(\mathrm{C});\mathrm{Z}) \arrow[u, "\wr"', "*_{n+1}"] \end{tikzcd}$$

where i_* is the group homomorphism of $\mathrm{H}_{2r}(\mathrm{Q}_n(\mathrm{C});\mathrm{Z})$ into $\mathrm{H}_{2r}(\mathrm{P}_{n+1}(\mathrm{C});\mathrm{Z})$ induced by the embedding $i:\mathrm{Q}_n(\mathrm{C}) \hookrightarrow \mathrm{P}_{n+1}(\mathrm{C})$, as follows.

Let $x_j \in \mathrm{H}^{2(n-r)}(\mathrm{Q}_n(\mathrm{C});\mathrm{Z})$ corresponding to the constructed bases for $\mathrm{H}_{2r}(\mathrm{Q}_n(\mathrm{C});\mathrm{Z})$ and let $h \in \mathrm{H}^2(\mathrm{P}_{n+1}(\mathrm{C});\mathrm{Z})$, then

$$\text{(I)}\qquad f^*(x_j) = \left.\begin{cases} h^{j+1} & \forall j \geq r+1 \\ 2\,h^{j+1} & j \leq r \end{cases}\right\} n = 2r+1,$$

$$\text{(II)}\qquad \left.\begin{array}{l} f^*(\ _j) = \begin{cases} h^{j+1} & \forall j \geq r+1 \\ 2\,h^{j+1} & \forall j \leq r-1 \end{cases} \\ f^*(x_r) = f^*(x'_r) = h^{r+1} \end{array}\right\} n = 2r.$$

Combining (I) and (II) we get

$$f^*(x_j) = \begin{cases} h^{j+1} & \forall j \geq r+1 \\ 2\,h^{j+1} & \forall j \leq r-1, \end{cases}$$

$$f^*(x_r) = h^{r+1},$$

$$(x'_r) = h^{r+1} \qquad \text{if} \quad n = 2r.$$

Furthermore,

$$\mathrm{Ker}\,(f^*) = \begin{cases} 0 & \text{if} \quad n = 2r+1 \\ x_r - x'_r & \text{if} \quad n = 2r. \end{cases}$$

BIBLIOGRAPHY

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[2] F. HIRZEBRUCH (1966) – *Topological methods in algebraic Geometry*. Springer, Berlin.