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## On Kähler manifolds and their generalizations

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**Geometria differenziale.** — *On Kähler manifolds and their generalizations.* Nota di GIOVANNI BATTISTA RIZZA (*), presentata (**) dal Corrisp. E. MARTINELLI.

RIASSUNTO. — Alcune condizioni formali legate alla struttura quasi complessa, da me introdotte nel 1965, e le loro « simmetriche » consentono di ottenere nuove caratterizzazioni delle varietà Kähleriane, delle varietà quasi Tachibana, delle varietà quasi Kotō e delle varietà Hermitiane.

Il lavoro contiene anche altri due risultati sulle varietà Kähleriane.

## 1. INTRODUCTION

Let V be a manifold with an almost complex structure J. In a paper of 1965 I considered for any tensor field of type $(1, 2)$ a set of conditions, essentially depending on the almost complex structure of V [1].

Now, if V is assumed to be an almost Hermite manifold, then those conditions and the "symmetric" ones can be used for the tensor field DJ obtained from J by covariant differentiation with respect to the Riemannian connection. This idea leads to characterize the Kähler manifolds, the almost Tachibana manifolds, the almost Kotō manifolds and the Hermite manifolds (Theorem 3, Theorem 4, Theorem 5, Theorem 6; Sec. 6).

It appears also a sort of symmetry between Kähler and almost Tachibana manifolds, as well as between almost Kotō and Hermite manifolds.

Finally, in Sec. 5 we show that if J is a recurrent field or if DJ is a symmetric field, then V is a Kähler manifold; and conversely (Theorem 1, Theorem 2).

## 2. ALMOST HERMITE MANIFOLDS

Let V be an *almost Hermite manifold* of dimension $2n$ and of class $C^{2n+1}$ [2].

Let $\mathscr{E}^r_s$ be the linear space of tensor fields of type $(r, s)$ on V. In particular, let $g$ be the symmetric field of $\mathscr{E}^0_2$ of class $C^1$, defining the Riemannian metric on V and let J be the field of $\mathscr{E}^1_1$ of class $C^{2n}$, defining the almost complex structure on V.

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(1) See n. [4] of the list of references.

(2) For the basic facts about almost Hermitian manifolds see K. YANO [8], Ch. 9; S. KOBAYASHI and K. NOMIZU [2], II, Ch. 9.

We denote by DJ the field of $\mathscr{C}_2^1$ obtained from J by covariant differentiation with respect to the Levi-Civita connection (Riemannian connection) [3]. Then from

$$c_2^1 (J \otimes J) = c_1^2 (J \otimes J) = - \delta , \tag{1}$$

it follows

$$c_3^1 (DJ \otimes J) + c_1^2 (J \otimes DJ) = 0 , \tag{2}$$

where $\delta$ is the classical Kronecker field of $\mathscr{C}_1^1$ and the symbol $c_j^i$ denotes contraction [4].

## 3. Isomorphisms and conditions in $\mathscr{C}_2^1$

Let $\sigma$, $\varepsilon$ be the homomorphisms of symmetry, of skew-symmetry of the linear space $\mathscr{C}_2^1$ [5]. Put $\alpha = \sigma - \varepsilon$.

The almost complex structure J of V leads to consider the *isomorphisms* $\lambda$, W defined for any field L of $\mathscr{C}_2^1$ by

$$\lambda L = c_3^1 (L \otimes J) , \tag{3}$$

$$WL = - c_2^2 (\lambda L \otimes J) \quad {}^{(6)\,(7)} . \tag{4}$$

It is worth remarking that [8]

$$\alpha\alpha = 1 \quad , \quad \lambda\lambda = -1 \quad , \quad WW = 1 , \tag{5}$$

$$\alpha\lambda = \lambda\alpha \quad , \quad \lambda W = W\lambda . \tag{6}$$

We are now able to write the *conditions*

$$\text{A)} \qquad WL = L \quad , \quad (\alpha W \alpha) L = L ,$$

$$\text{B)} \qquad (W\alpha) L = L \quad , \quad (\alpha W) L = L ,$$

$$\text{C)} \qquad (\alpha W \alpha W) L = L ,$$

$$\text{D)} \qquad (W \alpha W) L = L .$$

(3) The index of covariant differentiation is assumed to be the first lower index.

(4) See e.g. N. Bourbaki [1], p. 45.

(5) In other words
$(\sigma L)(X, Y) = \frac{1}{2}(L(X, Y) + L(Y, X))$, $(\varepsilon L)(X, Y) = \frac{1}{2}(L(X, Y) - L(Y, X))$, for any L of $\mathscr{C}_2^1$ and for any X, Y of $\mathscr{C}_0^1$.

(6) G. B. Rizza [5], p. 11, 13.

(7) Equivalent definitions of $\lambda$, W are
$(\lambda L)(X, Y) = JL(X, Y)$, $(WL)(X, Y) = - JL(X, JY)$, for any X, Y of $\mathscr{C}_0^1$. Here J is regarded as homomorphism of $\mathscr{C}_0^1$.

(8) G. B. Rizza [5], p. 11, 14.

Conditions A , B , C , D, which depends essentially on the almost complex structure of V, were first introduced in 1965 [9].

Likewise we also consider the *conditions*

$$\bar{A}) \qquad WL = -L \quad , \quad (\alpha W \alpha) L = -L ,$$

$$\bar{B}) \qquad (W\alpha) L = -L \quad , \quad (\alpha W) L = -L ,$$

$$\bar{C}) \qquad (\alpha W \alpha W) L = -L ,$$

$$\bar{D}) \qquad (W \alpha W) L = -L .$$

The first one of the conditions $\bar{A}$, denoted by $\bar{A}_1$, is known from the literature. More explicity, we have the Proposition

$P_1$ — $DJ$ *satisfies condition* $\bar{A}_1$ [10].

## 4. KÄHLER MANIFOLDS AND GENERALIZATIONS

We recall now that V is a *Kähler manifold*, if and only if $DJ = 0$ [11].

More generally, V is an *almost Tachibana manifold*, if and only if $\sigma DJ = 0$ [12].

V is called an *almost Kotō manifold*, if and only if we have $(1 + \alpha W \alpha)\, DJ = 0$ [13].

Finally, V is a *Hermite manifold*, if and only if $N = \frac{1}{2} \lambda \varepsilon W \varepsilon DJ = 0$ [14]. An equivalent condition is $(1 - \alpha W \alpha) DJ = 0$ [15].

The four classes of manifolds defined by the above conditions are well known in the literature. The most intersting class is that of the Kähler manifolds. The other classes are generalizations of the first one [16].

## 5. MANIFOLDS WITH A RECURRENT J OR A SYMMETRIC DJ

We are now able to state the following theorems

THEOREM 1. *If* J *is a recurrent tensor field, then* V *is a Kähler manifold; and conversely.*

(9) G. B. RIZZA [4], p. 240; see also G. B. RIZZA [5], p. 12.

(10) Proposition $P_1$ follows easily from (2), (1) of Sec. 2 and holds true for a general affine connection. See V. MANGIONE [3], p. 145 and G. B. RIZZA [6], p. 169.

(11) See e.g. K. YANO [8], (1.3), p. 70.

(12) See K. YANO [8], (1.6), p. 176.

(13) It is easy to prove that our condition is equivalent to condition (4.1) of K. YANO [8], p. 197. The almost Kotō manifolds are also called $O^*$ -spaces.

(14) See K. YANO [8], p. 63, 121 and (4.7), p. 127; see also G. B. RIZZA [5], (18), (21), pp. 14–15 and (4), p. 237.

(15) Compare with K. YANO [8], (2.6), p. 193.

(16) See K. YANO [8]; S. KOBAYASHI and K. NOMIZU [2], II.

THEOREM 2. *If* D J *is a symmetric tensor field, then* V *is a Kähler manifold; and conversely.*

The proof of Theorem 1 is easy. If J is a *recurrent tensor field* of $\mathscr{C}_1^1$, there exists a covariant vector field $a$ such that $\mathrm{DJ} = a \otimes \mathrm{J}$ (17). By virtue of (1) it follows

$$c_2^1 c_3^1 (\mathrm{DJ} \otimes \mathrm{J}) = c_2^1 c_3^1 (a \otimes \mathrm{J} \otimes \mathrm{J}) = -c_2^1 (a \otimes \delta) = -2na .$$

On the other hand, from (2) we get $c_2^1 c_3^1 (\mathrm{DJ} \otimes \mathrm{J}) = 0$. Therefore $a = 0$; then $\mathrm{DJ} = 0$ and V is a Kähler manifold. The converse is trivial ($a = 0$).

To prove Theorem 2, remark that if D J is a *symmetric tensor field* of $\mathscr{C}_2^1$, then $\varepsilon \mathrm{DJ} = 0$ and $\alpha \mathrm{DJ} = \mathrm{DJ}$. By virtue of Proposition $\mathrm{P}_1$, we have

$$(1 + \alpha \mathrm{W} \alpha)\, \mathrm{DJ} = (1 + \alpha \mathrm{W})\, \mathrm{DJ} = (1 - \alpha)\, \mathrm{DJ} = 2\, \varepsilon \mathrm{DJ} = 0.$$

Thus V is an almost Kotō manifold (Sec. 4). On the other hand, if $\varepsilon \mathrm{DJ} = 0$, then $\mathrm{N} = 0$ and V is a Hermite manifold; therefore $(1 - \alpha \mathrm{W} \alpha)\, \mathrm{DJ} = 0$ (Sec. 4). It is now easy to conclude that $\mathrm{DJ} = 0$; thus V is a Kähler manifold. The converse is trivial.

Note that the converse paits of Theorem 1 and of Theorem 2 are still true if V is assumed to be of class $\mathrm{C}^2$ and J of class $\mathrm{C}^1$.

## 6. CHARACTERIZATION THEOREMS

The assumption that D J satisfies one of the conditions of Sec. 3 leads to Proposition $\mathrm{P}_1$ of Sec. 3 and to four theorems. Namely

THEOREM 3. *If* D J *satisfies one of the conditions* $\mathrm{A}_1 , \overline{\mathrm{B}}_1 , \overline{\mathrm{B}}_2 , \mathrm{D}$, *then* V *is a Kähler manifold: and conversely.*

THEOREM 4. *If* D J *satisfies one of the conditions* $\mathrm{B}_1 , \mathrm{B}_2 , \overline{\mathrm{D}}$, *then* V *is an almost Tachibana manifold; and conversely.*

THEOREM 5. *If* D J *satifies one of the conditions* $\overline{\mathrm{A}}_2 , \mathrm{C}$, *then is an almost Kotō manifold; and conversely.*

THEOREM 6. *If* D J *satisfies one of the conditions* $\mathrm{A}_2 , \overline{\mathrm{C}}$, *then is a Hermite manifold; and conversely.*

If you regard A , B , C , D and $\overline{\mathrm{A}} , \overline{\mathrm{B}} , \overline{\mathrm{C}} , \overline{\mathrm{D}}$ (Sec. 3) as two sets of " symmetric " conditions, and take into account Proposition $\mathrm{P}_1$, then by Theorem 3, Theorem 4 Kähler and almost Tachibana manifolds appear as " symmetric " manifolds.

The same " symmetry " links almost Kotō and Hermite manifolds, because of Theorem 5, Theorem 6.

(17) T. J. WILLMORE [7], p. 245.

It will appeai from the proofs that Theorem 4, Theorem 5, Theoiem 6 and the converse part of Theorem 3 hold true if V is assumed to be of class $C^2$ and J of class $C^1$.

## 7. PROOFS

To prove Theorem 3, remark first that the case $A_1$ is trivial by virtue of Proposition $P_1$. Moreover, from (5) and $P_1$ it follows that each one of conditions $\overline{B}_1$, D for the tensor field DJ is equivalent to condition $\overline{B}_2$. Now since we have

$$(1 + \alpha W)\, DJ = (1 - \alpha)\, DJ = 2\,\varepsilon\, DJ,$$

Theorem 2 of Sec. 5 leads us to the conclusion in the cases $\overline{B}_1$, $\overline{B}_2$, D. Therefore the proof of Theorem 3 is complete.

From (5) and Proposition $P_1$ it follows that each one of conditions $B_1$, $\overline{D}$ for DJ is equivalent to condition $B_2$. This proves Theorem 4, since we have

$$(1 - \alpha W)\, DJ = (1 + \alpha)\, DJ = 2\,\sigma\, DJ\,.$$

Finally, by virtue of Proposition $P_1$, conditions C, $\overline{C}$ for the tensor field DJ are equivalent to conditions $\overline{A}_2$, $A_2$, iespectively. Therefore the proofs of Theorem 5, Theorem 6 are trivial.

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