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# Rendiconti

Awdhesh Kumar

## On recurrence vectors in a $F_n^*$-space

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**Geometria differenziale.** — *On recurrence vectors in a* $F_n^*$*-space.* Nota di AWDHESH KUMAR, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si studiano i vettori ricorrenti $\beta_s$ in uno spazio di Finsler ricorrente al primo ordine $F_n^*$, e si assegnano condizioni sufficienti per la simmetria del tensore $\beta_{s;h}$.

## 1. INTRODUCTION

Let us consider an $n$-dimensional affinely connected Finsler space $F_n$ [1] (1) having symmetric connection coefficient $\Gamma^i_{jk}(x, \xi)$. The covariant derivative of any vector field $X^i$ with respect $\Gamma^i_{jk}(x, \xi)$ is given by

$$X^i_{;k} = \partial_k X^i + X^s \Gamma^i_{sk} + \dot\partial_h X^i \partial_k \xi^h. \tag{1.1}$$

The well known commutation formula involving the above covariant derivative is given by

$$2\, X^i_{;[hk]} = X^s \tilde K^i_{shk}, \tag{1.2}$$

where

$$\tilde K^i_{hjk}(x, \xi) \overset{\text{def.}}{=} 2\,\{\partial_{[k} \Gamma^i_{j]h} + \dot\partial_s \Gamma^i_{h[j} \partial_{k]} \xi^s + \Gamma^i_{s[k} \Gamma^s_{j]h}\} \tag{1.3}$$

is called relative curvature tensor field and satisfies the following identities [1]:

$$\tilde K^i_{hjk;s} + \tilde K^i_{hks;j} + \tilde K^i_{hsj;k} = 0, \tag{1.4}$$

$$\tilde K^i_{hjk} + \tilde K^i_{jkh} + \tilde K^i_{khj} = 0, \tag{1.5}$$

$$a)\ \tilde K^i_{hji} = \tilde K_{hj}, \qquad b)\ \tilde K^i_{ihj} = \tilde K_{hj} - \tilde K_{jh} \tag{1.6}$$

and

$$\tilde K^i_{hjk} = -\tilde K^i_{hkj}. \tag{1.7}$$

In an $F_n$, if the relative curvature tensor field $\tilde K^i_{hjk}(x, \xi)$ satisfies the relation:

$$\tilde K^i_{hjk;s} = \beta_s \tilde K^i_{hjk}, \tag{1.8}$$

where $\beta_s(x)$ is any vector field, then the space is called recurrent Finsler space of first order or $F_n^*$-space and $\beta_s$ is known as recurrence vector.

(*) Nella seduta del 16 aprile 1977.
(1) The numbers in brackets refer to the references given at the end of the paper.
(2) $2\, A_{[hk]} = A_{hk} - A_{kh}$.
(3) $\dot\partial_i \equiv \partial/\partial \dot x^i$ and $\partial_i \equiv \partial/\partial x^i$.

## 2. Decomposition of $\tilde{K}^{i}_{hjk}(x, \xi)$

In view of the basic condition (1.8), the Bianchi's identity (1.4) for the relative curvature tensor field $\tilde{K}^{i}_{hjk}(x, \xi)$ reduces to

$$\beta_s \tilde{K}^{i}_{hjk} + \beta_j \tilde{K}^{i}_{hks} + \beta_k \tilde{K}^{i}_{hsj} = 0 . \tag{2.1}$$

Contracting the last formula by $v^s$ and summing over the index $s$, we can get

$$\tilde{K}^{i}_{hjk} = \beta_k E^{i}_{hj} - \beta_j E^{i}_{hk} , \tag{2.2}$$

where

$$E^{i}_{hj} \overset{\text{def.}}{=} \tilde{K}^{i}_{hjs} v^s \tag{2.3}$$

and

$$\beta_s v^s = 1 . \tag{2.4}$$

By virtue of equations (1.6 *b*), (1.7) and (2.3), we can deduce

$$a) \quad E^{i}_{hj} v^j = 0 \qquad \text{and} \qquad b) \quad E^{i}_{ij} = 0 . \tag{2.5}$$

Now, if $E^{*i}_{hj}$ is any tensor satisfying

$$\tilde{K}^{i}_{hjk} = \beta_k E^{*i}_{hj} - \beta_j E^{*i}_{hk} . \tag{2.6}$$

then by subtracting the above relation from (2.2), we find

$$\beta_k (E^{*i}_{hj} - E^{i}_{hj}) = \beta_j (E^{*i}_{hk} - E^{i}_{hk}) . \tag{2.7}$$

In view of the last equation we can introduce a tensor $Q^{i}_{h}$:

$$E^{*i}_{hj} = E^{i}_{hj} + \beta_j Q^{i}_{h} . \tag{2.8}$$

Conversely, if $E^{i}_{hj}$ satisfies (2.2) and $Q^{i}_{h}$ is any tensor, then the tensor $E^{*i}_{hj}$ satisfies (2.6). Such a tensor $E^{*i}_{hj}$ of the form (2.8) may be regarded as a *symmetric* one. We shall prove this fact. For this purpose, let us introduce formula (2.2) into the left-hand side of the Bianchi identity (1.5); we obtain

$$\beta_k (E^{i}_{hj} - E^{i}_{jh}) + \beta_h (E^{i}_{jk} - E^{i}_{kj}) + \beta_j (E^{i}_{hk} - E^{i}_{kh}) = 0. \tag{2.9}$$

Now, transvecting this formula by $v^k$ and taking care of the equation (2.4), we have

$$E^{i}_{hj} - E^{i}_{jh} = \beta_h Q^{i}_{j} - \beta_j Q^{i}_{h} . \tag{2.10}$$

i.e.

$$E^{i}_{hj} + \beta_j Q^{i}_{h} = E^{i}_{jh} + \beta_h Q^{i}_{j} , \tag{2.11}$$

where we have put

$$Q_j^i \equiv (E_{hj}^i - E_{jh}^i)\, v^h. \tag{2.12}$$

On the other hand, with the help of the equations (2.3) and (2.5 *a*), the above formula takes the form

$$Q_j^i = E_{hj}^i\, v^h = \tilde{K}_{hjs}^i\, v^s\, v^h. \tag{2.13}$$

Now, by virtue of the equations (2.8) and (2.10), we can conclude that

$$E_{hj}^{*i} = E_{jh}^{*i}. \tag{2.14}$$

## 3. DISCUSSION

Contracting the formula (2.6) with respect to the indices $i$ and $k$, and using the relation (1.6 *a*), we get

$$\tilde{K}_{hj} = \beta_i\, E_{hj}^{*i} - \beta_j\, E_{hi}^{*i}. \tag{3.1}$$

Let us suppose the symmetry of $\tilde{K}_{hj}$, then, from the above result, we obtain

$$\beta_j\, E_{hi}^{*i} = \beta_h\, E_{ji}^{*i}. \tag{3.2}$$

From the last formula, we get

$$E_{hi}^{*i} = \alpha\beta_h. \tag{3.3}$$

By virtue of the equations (2.5 *b*), (2.8) and (2.14), we deduce

$$E_{hi}^{*i} = E_{ih}^{*i} = E_{ih}^i + \beta_h\, Q_i^i. \tag{3.4}$$

Comparing the above result with the equation (3.3), we find

$$\alpha = Q_i^i. \tag{3.5}$$

In view of equations (1.6 *a*), (1.7) and (2.13), the last result gives

$$\alpha = Q_i^i = E_{hi}^i\, v^h = (-\tilde{K}_{hs}\, v^s\, v^h), \tag{3.6}$$

i.e.

$$\tilde{K}_{hs}\, v^s\, v^h = -\alpha. \tag{3.7}$$

In view of the basic condition (1.8) and of formula (2.13), we have

$$\beta_j\, Q_h^i = \beta_j\, \tilde{K}_{hks}^i\, v^s\, v^k = (\tilde{K}_{khs;j}^i)\, v^s\, v^k. \tag{3.8}$$

Now, with the help of the equations (1.7) and (3.8), we obtain

$$\beta_j\, Q_h^i - \beta_h\, Q_j^i = (\tilde{K}_{khs;j}^i + \tilde{K}_{ksj;h}^i)\, v^s\, v^k. \tag{3.9}$$

By virtue of the identities (1.4), (1.7) and (2.4), the above relation yields

$$\beta_j \, Q^i_h - \beta_h \, Q^i_j = - \tilde{K}^i_{kjh;s} \, v^s \, v^k = \beta_s \, v^s \, \tilde{K}^i_{khj} \, v^k = \tilde{K}^i_{khj} \, v^k. \tag{3.10}$$

Now, on the other hand, let us consider the basic definition (1.8) as a differential equation of $\tilde{K}^i_{hjk}(x, \xi)$. Then its integrability condition takes the form:

$$H_{ms} \, \tilde{K}^i_{hjk} = \tilde{K}^a_{hjk} \, \tilde{K}^i_{asm} - \tilde{K}^i_{ajk} \, \tilde{K}^a_{hsm} - \tilde{K}^i_{hak} \, \tilde{K}^a_{jsm} - \tilde{K}^i_{hja} \, \tilde{K}^a_{ksm}, \tag{3.11}$$

where we have put

$$H_{ms} \overset{\text{def.}}{=} (\beta_{s;m} - \beta_{m;s}) . \tag{3.12}$$

Contracting (3.11) with respect to the indices $i$ and $k$ and taking care of the equation (1.6 $a$), we obtain

$$H_{ms} \, \tilde{K}_{hj} = - \tilde{K}_{aj} \, \tilde{K}^a_{hsm} - \tilde{K}_{ha} \, \tilde{K}^a_{jsm} . \tag{3.13}$$

Transvecting the last result by $v^h \, v^j$ and using equation (3.7) and the symmetric property of the Ricci–tensor $\tilde{K}_{hj}$ (i.e. $\tilde{K}_{hj} = \tilde{K}_{jh}$), we find

$$\alpha H_{ms} = 2 \, \tilde{K}^a_{hsm} \, v^h \, (\tilde{K}_{aj} \, v^j) . \tag{3.14}$$

Next, introducing formula (3.10) into the right-hand side of the above relation, we get

$$\alpha H_{ms} = 2 \, (\beta_m \, Q^a_s - \beta_s \, Q^a_m) \, (\tilde{K}_{aj} \, v^j) . \tag{3.15}$$

Tranvecting (2.12) by $v^j$ and taking notice of the equation (2.5 $a$), we obtain

$$Q^i_j \, v^j = 0 . \tag{3.16}$$

Thus, contracting the formula (3.15) by $v^m$ and noting equations (2.4) and (3.16), we get

$$\alpha H_{ms} \, v^m = 2 \, Q^a_s \, \tilde{K}_{aj} \, v^j, \tag{3.17}$$

or

$$\alpha \beta_h \, H_{ms} \, v^m = 2 \, Q^a_s \, \beta_h \, \tilde{K}_{aj} \, v^j. \tag{3.18}$$

On an analogous way, we have

$$\alpha \beta_s \, H_{hm} \, v^m = - 2 \, Q^a_h \, \beta_s \, \tilde{K}_{aj} \, v^j, \tag{3.19}$$

where we have used the skew symmetric property of $H_{hm}$, i.e.

$$H_{hm} = - H_{mh} . \tag{3.20}$$

Adding the two equation (3.18) and (3.19) side by side, we get

$$\alpha \, (\beta_h \, H_{ms} \, v^m + \beta_s \, H_{hm} \, v^m) = \beta H_{hs} , \tag{3.21}$$

where we have used (3.15).

In view of the equation (3.20) and the non-vanishing property of the function $\alpha(x)$, the above formula reduces to

$$H_{hs} = \beta_h H_{ms} v^m - \beta_s H_{mh} v^m. \tag{3.22}$$

Consequently, from the above equation it is clear that when, and only when, we have

$$\beta_h H_{ms} v^m = \beta_s H_{mh} v^m, \tag{3.23}$$

we find

$$H_{hs} = 0 \tag{3.24}$$

or

$$\beta_{s;h} = \beta_{h;s}. \tag{3.25}$$

Transvecting formula (3.23) by $v^h$ and using the relation (2.4), we get

$$H_{ms} v^m = 0, \tag{3.26}$$

where we have also used

$$H_{mh} v^m v^h = 0. \tag{3.27}$$

Conversely, from (3.26) we get (3.22). Hence (3.22) and (3.25) are equivalent to each other. Thus we can state the following:

CONCLUSION (3.1). *In an* $F_n^*$ *for which the Ricci tensor* $\tilde{K}_{hj}(x, \xi)$ *is symmetric, the tensor* $\beta_{h;s}$ *is symmetric when, and only when, there exists a contravariant vector* $v^i(x)$ *satisfying* (2.4),

$$\tilde{K}_{hj} v^h v^j \neq 0 \quad \text{and} \quad (\beta_{h;s} - \beta_{s;h}) v^s = 0.$$

## 4. REMARK

Let us consider the case of $\alpha = 0$. According to (3.21), such a case occurs when we have $(\beta_h H_{ms} v^m + \beta_s H_{hm} v^m - H_{hs}) \neq 0$.

Now, transvecting formula (3.13) by $v^m$ side by side and summing over the index $m$, we get

$$H_{ms} v^m \tilde{K}_{hj} = - \tilde{K}_{aj} \tilde{K}^a_{hsm} v^m - \tilde{K}_{ha} \tilde{K}^a_{jsm} v^m. \tag{4.1}$$

Next, multiplying the last result by $\beta_k$ and using the fundamental condition (1.8), we have

$$\beta_k H_{ms} v^m \tilde{K}_{hj} = - (\tilde{K}^a_{hsm;k}) \tilde{K}_{aj} v^m - (\tilde{K}^a_{jsm;k}) v^m \tilde{K}_{ha}. \tag{4.2}$$

In and analogous way we find

$$\beta_s \mathrm{H}_{km} v^m \tilde{\mathrm{K}}_{hj} = (\tilde{\mathrm{K}}^a_{hkm;s}) \tilde{\mathrm{K}}_{aj} v^m + (\tilde{\mathrm{K}}_{jkm;s}) v^m \tilde{\mathrm{K}}_{ha}, \tag{4.3}$$

where we have used (3.20).

Adding the last two relations side by side, we obtain

$$(\beta_k \mathrm{H}_{ms} - \beta_s \mathrm{H}_{mk}) v^m \tilde{\mathrm{K}}_{hj} = \tilde{\mathrm{K}}^a_{hsk} \tilde{\mathrm{K}}_{aj} - \tilde{\mathrm{K}}^a_{jsk} \tilde{\mathrm{K}}_{ha}, \tag{4.4}$$

where we have also used (2.4) and (3.20).

In view of equation (3.13), the above relation yields

$$(\beta_k \mathrm{H}_{ms} - \beta_s \mathrm{H}_{mk}) v^m \tilde{\mathrm{K}}_{hj} = \mathrm{H}_{ks} \tilde{\mathrm{K}}_{hj}, \tag{4.5}$$

i.e.

$$\{\mathrm{H}_{ks} - (\beta_k \mathrm{H}_{ms} - \beta_s \mathrm{H}_{mk}) v^m\} \tilde{\mathrm{K}}_{hj} = 0 . \tag{4.6}$$

from which, by assumption, we get

$$\tilde{\mathrm{K}}_{hj} = 0 . \tag{4.7}$$

This is absurd. Consequently, such a case of $\alpha = 0$ is not relevant to the subject.

## 5. Another Conclusion

In § 3, we have introduced a proportional factor $\alpha$ and we have obtained (3.7). And, by use of this factor $\alpha$, we have derived (3.21) from which, under $\alpha \neq 0$, we have found the condition for the symmetry of $\beta_{h;s}$.

However, as we have mentioned in § 4, from (3.13) we have derived (4.6) straightly. Here we have to take care of the fact that, having no-connection with $\tilde{\mathrm{K}}_{hj} = \tilde{\mathrm{K}}_{jh}$, the formula (4.6) has been derived from (3.13). Furthermore, (3.13) has not yet any connection with the symmetry of $\tilde{\mathrm{K}}_{hj}$, but it has not been derived from the defining equation of the space itself. In this case, assuming $\tilde{\mathrm{K}}_{hj} \neq 0$, we have (3.22). Thus we see, by use of (2.4), the condition for the gradient of $\beta_h$ as $\mathrm{H}_{hk} v^h = 0$.

Considering these facts, we can state that the existence of non-vanishing $\alpha$ is related to the symmetry assumption of $\tilde{\mathrm{K}}_{hj}$. According to these facts, we are able to get the following further conclusion, having no-connection with $\tilde{\mathrm{K}}_{hj} = \tilde{\mathrm{K}}_{jh}$.

CONCLUSION (5.1). *In a recurrent Finsler space of first order, with a non-vanishing Ricci tensor, the recurrence vector* $\beta_h$ *is a gradient one, when and only when there exists a contravariant vector* $v^i$ *satisfying* (2.4) *and* $(\beta_{h;s} - \beta_{s;h}) v^s = 0$.

## 6. CONTINUATION OF DISCUSSION

In order to get the second supposition, i.e. (3.7), in the conclusion of § 3, it is sufficient to take into consideration

$$\tilde{K}_{hj}\, v^j = -\alpha\beta_h\,, \qquad (\alpha \neq 0) \tag{6.1}$$

because contraction of the above formula by $v^h$ gives (3.7).

In view of the equations (1.6 *a*), (3.6) and (3.10), we conclude that

$$\tilde{K}_{jh}\, v^j = -\alpha\beta_h - \beta_j\, Q^j_h\,. \tag{6.2}$$

Now, comparing the two equations (6.1) and (6.2), we get

$$\beta_j\, Q^j_h = 0\,. \tag{6.3}$$

Thus, transvecting formula (6.1) by $Q^h_s$ and taking care of the above relation, we have

$$\tilde{K}_{hj}\, Q^h_s\, v^j = 0\,. \tag{6.4}$$

On the other hand, with the help of the formula (3.17) the last equation takes the form

$$\alpha\tilde{K}_{ks}\, v^k = 0\,. \tag{6.5}$$

Since $\alpha \neq 0$, the last formula can be rewritten as

$$\tilde{K}_{ks}\, v^k = 0\,. \tag{6.6}$$

This is the third supposition in the conclusion of § 3. Thus, the conclusion in § 3 may be replaced by the following:

CONCLUSION. *In a recurrent Finsler space of first order, where the Ricci tensor* $\tilde{K}_{hj}(x, \dot{x})$ *is symmetric, if we choose a contravariant vector* $v^i$ *so as to satisfy* (2.4) *and* (6.1), *where* $\alpha$ *means a scalar function, then* $\beta_{h;s}$ *becomes a symmetric tensor.*

It is easy to see that (6.1) is equivalent to (6.3). In fact, as we always have (6.2), in view of (6.3) we get (6.1) and conversely.

On the other hand, multiplying formula (3.10) by $\beta_i$ and summing over the index $i$, we find

$$\beta_i\,(\beta_j\, Q^i_h - \beta_h\, Q^i_j) = \beta_i\, \tilde{K}^i_{khj}\, v^k. \tag{6.7}$$

In view of (6.3), the last formula yields

$$\beta_i\, \tilde{K}^i_{khj}\, v^k = 0\,, \tag{6.8}$$

or

$$(\tilde{K}^i_{khj;i})\, v^k = 0\,. \tag{6.9}$$

Now, with the help of the equations (1.4), (1.6) and (1.8), the above result takes the form

$$\beta_h \tilde{K}_{kj} v^k = \beta_j \tilde{K}_{kh} v^k. \tag{6.10}$$

On the other hand, transvecting the above formula by $v^h$ and taking care of equations (2.4) and (3.7), we get

$$\tilde{K}_{kj} v^k = -\alpha\beta_j . \tag{6.11}$$

Namely, from (6.8) we get (6.1), and consequently we have (6.3). By this reason, (6.8) is equivalent to (6.3), i.e. (6.1). Accordingly (6.1) can be replaced by (6.8). Thus, from the above conclusion we obtain the following:

THEOREM (6.1). *In a recurrent Finsler space of first order, where the Ricci tensor is symmetric, in order to get* $\beta_{h;s} = \beta_{s;h}$ *it is sufficient that there exists a contravariant vector* $v^i$ *satisfying* (2.4) *and* (6.8).

## REFERENCES

[1] A. G. WALKER (1956) - *On Ruse's Spaces of recurrent curvature*, « Proc. London Math. Soc. », *52* (2), 36-54.

[2] H. RUND (1959) - *The differential geometry of Finsler spaces*, Springer-Verlag, Berlin.

[3] K. TAKANO (1964) - *On Y. C. Wong's Conjecture*, « Tensor, N.S. », *15* (3), 175-180.

[4] Y. C. WONG and K. YANO (1961) - *Projectly flat spaces with recurrent curvature*, « Comment. Math. Hélv. », *35*, 223-232.