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Remark on nonoscillatory solution of nonlinear n-th order functional differential equation with deviating arguments

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Equazioni funzionali. — *Remark on nonoscillatory solution of nonlinear n–th order functional differential equation with deviating arguments* (*). Nota di LU–SAN CHEN e CHEH–CHIH YEH, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si dà una condizione necessaria perché le soluzioni di un'equazione differenziale funzionale non lineare di ordine n, con argomento ritardato, siano nonoscillatorie e limitate.

Nonoscillatory theorems are presented for the nonlinear n-th order functional differential equation with deviating arguments

$$L_n x(t) + (-1)^n p(t) f(x[g_1(t)], \cdots, x[g_m(t)]) = 0$$

where L_n is an operator defined recursively by

$$L_0 x(t) = x(t) \quad , \quad L_i x(t) = \frac{1}{r_i(t)} \frac{d}{dt} L_{i-1} x(t) \, , r_i(t) > 0 \, ,$$

$i = 1, 2, \cdots, n$, $r_n(t) = 1$, and

$$\int^{\infty} r_i(t) \, dt = \infty \, , \qquad i = 1, 2, \cdots, n-1.$$

1. INTRODUCTION

In this paper we are concerned with the following nonlinear n-th order functional differential equation with deviating arguments

$$(*) \qquad L_n x(t) + (-1)^n p(t) f(x[g_1(t)], \cdots, x[g_m(t)]) = 0 .$$

For this equation the following conditions are assume to hold throughout the paper:

(α) $r_i(t)$ *and* $p(t)$ *are continuous and positive on* $[t_0, \infty)$ *and*

$$\int^{\infty} r_i(t) \, dt = \infty \, , \qquad i = 1, 2, \cdots, n-1,$$

(β) $f(y_1, \cdots, y_m)$ *is continuous on* R^m *and if* $y_j > 0$ *for every* $j = 1, 2, \cdots, m$, *then* $f(y_1, \cdots, y_m) > 0$, *and if* $y_j < 0$ *for every* $j = 1, 2, \cdots, m$, *then* $f(y_1, \cdots, y_m) < 0$;

(γ) $g_j(t)$ *is continuous on* $[t_0, \infty)$ *and* $\lim_{t \to \infty} g_j(t) = \infty$, $j = 1, 2, \cdots, m$.

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The purpose of this paper is to establish the necessary and/or sufficient conditions for equation (*) to have a bounded nonoscillatory solution. Results in this respect can be found in [2, 3, 5]. In order to obtain our main results, we need the following Kirguradze's lemma [4] which was extended by the Authors [1].

LEMMA. *Let $u(t)$ be a positive n–times continuously differentiable function on an interval $[a, \infty)$. If $L_n u(t)$ is of constant sign and not identically zero for all large t, then there exists a $t_u \geq a$ and an integer k, $0 \leq k \leq n$ with $n+k$ odd if $L_n u(t) \leq 0$, $n+k$ even if $L_n u(t) \geq 0$ and such that for every $t \geq t_u$.*

$$\begin{cases} L_\nu u(t) \geq 0, & \nu = 0, 1, \cdots, k-1 \\ (-1)^{n+\nu} L_\nu u(t) \geq 0, & \nu = k, k+1, \cdots, n. \end{cases}$$

2. MAIN RESULTS

THEOREM 1. *A necessary condition for equation (*) to have a bounded nonoscillatory solution is*

$$(1) \qquad \int^{\infty} r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \int_{s_{n-1}}^{\infty} p(s)\, ds\, ds_{n-1} \cdots ds_1 < \infty .$$

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (*). Without loss of generality, we may assume that $x(t) > 0$ for $t \geq T$, where T is large enough. By condition (γ), we obtain $x[g_j(t)] > 0$, $j = 1, 2, \cdots, m$ for $t \geq T$. It follows from (*), (α) and (β) that

$$L_n x(t) \begin{cases} < 0, & n \text{ even} \\ > 0, & n \text{ odd}. \end{cases}$$

The lemma implies $(-1)^{i+1} L_i x(t) \geq 0$, $i = 1, 2, \cdots, n-1$ eventually. Thus, $x(t)$ is a nondecreasing function and since $x(t) > 0$ and is bounded, we have $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x[g_j(t)] = l > 0$, $j = 1, 2, \cdots, m$. Using (β) we may choose $T_1 \geq T$ such that for $t \geq T_1$

$$f(x[g_1(t)], \cdots, x[g_m(t)]) \geq \tfrac{1}{2} f(l, \cdots, l) = M .$$

From (*) we have

$$(2) \qquad (-1)^n L_{n-1} x(t) \geq \int_t^{\infty} p(s) f(x[g_1(s)], \cdots, x[g_m(s)])\, ds \geq M \int_t^{\infty} p(s)\, ds .$$

Integrating (2), $n-2$ times from t to ∞, we have

$$(3) \qquad L_1 x(t) \geq M \int_t^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \int_{s_{n-1}}^{\infty} p(s)\, ds\, ds_{n-1} \cdots ds_2 .$$

Hence for $t \geq u \geq T_1$ we obtain

$$x(t) > x(t) - x(u) \geq \tag{4}$$

$$\geq M \int_u^t r_1(s_1) \int_{s_1}^{\infty} r_2(s_2) \cdots \int_{s_{n-2}}^{\infty} r_{n-1}(s_{n-1}) \int_{s_{n-1}}^{\infty} p(s)\, ds\, ds_{n-1} \cdots ds_1 .$$

Letting $t \to \infty$ in (4), we complete our proof.

We shall now clarify the importance of Theorem 1 by applying it to the particular cases:

$$\left[\frac{1}{r_{n-k}(t)} x^{(n-k)}(t)\right]^{(k)} + (-1)^n p(t) f(x[g_1(t)], \cdots, x[g_m(t)]) = 0, \tag{**}$$

and

$$x^{(n)}(t) + (-1)^n p(t) f(x[g_1(t)], \cdots, x[g_m(t)]) = 0 . \tag{***}$$

COROLLARY 1. *A necessary condition for equation* (**) *to have a bounded nonoscillatory solution is*

$$\int^{\infty} r_{n-k}(t)\, t^{n-k-1} \int_t^{\infty} (s-t)^{k-1} p(s)\, ds\, dt < \infty . \tag{5}$$

COROLLARY 2. *A necessary condition for equation* (***) *to have a bounded nonoscillatory solution is*

$$\int^{\infty} t^{n-1} p(t)\, dt < \infty . \tag{6}$$

REMARK. By taking n even and $m = 1$ in Corollary 2, we obtain Grefsrud's result [2, Theorem 1].

We assume that under the initial conditions $x(t) = \varphi(t)$, $t \leq t_0$, $x^{(k)}(t_0) = x_0^{(k)}$, $k = 1, 2, \cdots, n-1$, the equation (***) has a solution which exists for all $t \geq t_0$. Next, we establish sufficient conditions for equation (***) to have a bounded nonoscillatory solution.

THEOREM 2. *Let condition* (6) *hold and* $f(y_1, \cdots, y_m)$ *be nondecreasing with respect to* $y_1, \cdots, y_m$. *Then there exists a bounded nonoscillatory solution of* (***).

Proof. We now define a sequence $\{x_k(t)\}_{k=0}^{\infty}$ by

$$\begin{cases} x_0(t) = 1 \\ x_{k+1}(t) = 1 + \displaystyle\int_{t_0}^{t} \frac{(s-t_0)^{n-1}}{(n-1)!} p(s) f(x_k[g_1(s)], \cdots, x_k[g_m(s)])\, ds \\ \qquad + \displaystyle\int_{\infty}^{\infty} \frac{(s-t_0)^{n-1} - (s-t)^{n-1}}{(n-1)!} p(s) f(x_k[g_1(s)], \cdots, x_k[g_m(s)])\, ds, \end{cases} \tag{7}$$

where t_0 is chosen such that

$$(8)\quad f(2,\cdots,2)\left\{\int_{t_0}^{t}\frac{(s-t_0)^{n-1}}{(n-1)!}\,p(s)\,\mathrm{d}s+\int_{t}^{\infty}\frac{(s-t_0)^{n-1}-(s-t)^{n-1}}{(n-1)!}\,p(s)\,\mathrm{d}s\right\}\leq 1\,.$$

By induction, we see easily that for $t \geq t_0$

$$(9)\qquad 1 \leq x_k(t) \leq x_{k+1}(t) \leq 2\,, \qquad k = 0\,,1\,,\cdots.$$

Hence the sequence $\{x_k(t)\}_{k=0}^{\infty}$ is nondecreasing and uniformly bounded on $[t_0\,,\infty)$. It follows from Lebesgue's monotone convergence theorem that there exists a $x(t)$ such that $x(t)=\lim\limits_{k\to\infty} x_k(t)$. Thus $1 \leq x(t) \leq 2\,, t \geq t_0$ and $x(t)$ is the solution of

$$(10)\qquad x(t) = 1 + \int_{t_0}^{t}\frac{(s-t_0)^{n-1}}{(n-1)!}\,p(s)\,f(x[g_1(s)]\,,\cdots,x[g_m(s)])\,\mathrm{d}s$$
$$+\int_{t}^{\infty}\frac{(s-t_0)^{n-1}-(s-t)^{n-1}}{(n-1)!}\,p(s)\,f(x[g_1(s)]\,,\cdots,x[g_m(s)])\,\mathrm{d}s\,.$$

Differentiating (10) n-times we have the equation (***). It means that (10) is a nonoscillatory solution of equation (***).

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