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A Note on the Subdivision Norm Infimum Function

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Analisi funzionale. — *A Note on the Subdivision Norm Infimum Function.* Nota di WAYNE C. BELL, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Supposto che S sia un insieme, F un campo di sottoinsiemi di S, μ una funzione additiva su F a valori reali non negativi e $p > 1$, esistono allora e sono uguali gli integrali $\int \mu^p, \int (\mu^*)^p$ dove μ^* è la funzione definita per V in F con la formula $\mu^*(V) = \inf(\max(\mu(I) / I \in D)$, essendo D una divisione di V.

1. INTRODUCTION

Suppose S is a set, F a field of subsets of S, μ an additive (on disjoint elements) function from F into the nonnegative reals, and "integral" means refinement limit of sums over finite subdivisions of S by elements of F (see [3]).

In [1] W. D. L. Appling proved that $\int \mu^2 = \int (\mu^*)^2$ where μ^* is the subdivision norm infimum function (section 2). Subsequently he asked whether the statement

$$\int \mu^p = \int (\mu^*)^p$$

is true for values of p other than 2. It is clear that it may not be if $p = 1$ and that for $0 < p < 1$ the integrals may not exist. We will show that, for $p > 1$, the statement holds and, consequently, the p order Lipschitz classes (section 3) of μ and $\int \mu^*$ are equal.

2. THE SUBDIVISION NORM INFIMUM FUNCTION

For each $V \in F$ define μ^* by $\mu^*(V) = \inf\{\max\{\mu(I) \mid I \in D\} \mid D$ is a subdivision of $V\}$ and notice that if D is a subdivision of S and $E \ll D$ (E is a refinement of D), then

$$\sum_D \mu^*(V) \leq \sum_E \mu^*(I) \leq \mu(S).$$

Consequently $\int \mu^*$ exists and

i) $$\mu^* \leq \int \mu^* \leq \mu.$$

If, in addition, $p > 1$, then

ii) $$\sum_E \mu(I)^p \leq \sum_D \mu(V)^p$$

and similarly for $\int \mu^*$ so that each of $\int \mu^p$ and $\int \left(\left(\int \mu^*\right)^p\right)$ exists.

(*) Nella seduta del 16 aprile 1977.

Our arguments will involve the following inequality. Its proof is by elementary methods and is, therefore, omitted.

Suppose $0 < a < b < 1/e$ and $p > 1$, then

iii) $$b^p - a^p < b - a .$$

For $K > 0$ we have $(K\mu)^* = K\mu^*$ and $\int (K\mu)^p = K^p \int \mu^p$ (if either exists) therefore we will henceforth assume, without loss of generality, that $\mu(S) < 1/e$.

LEMMA 2.1. *If $p > 1$, then $\int (\mu^*)^p$ exists and is $\int \left(\int \mu^*\right)^p$.*

Proof. Suppose $c > 0$ and $H \in F$. Let D be a subdivision of H such that if $E \ll D$, then

$$\left| \int_H \left(\int_V \mu^*(I) \right)^p - \sum_E \left(\int_V \mu^*(I) \right)^p \right| < c/2$$

and

$$\left| \int_H \mu^*(I) - \sum_E \mu^*(I) \right| < c/2 .$$

If $E \ll D$, then we have

$$\left| \int_H \left(\int_V \mu^*(I) \right)^p - \sum_E \mu^*(V)^p \right| \leq \left| \int_H \left(\int_V \mu^*(I) \right)^p - \right.$$

$$\left. - \sum_E \left(\int_V \mu^*(I) \right)^p \right| + \left| \sum_E \left(\int_V \mu^*(I) \right)^p - \sum_E \mu^*(V)^p \right|$$

which by i) is

$$\leq c/2 + \sum_E \left[\left(\int_V \mu^*(I) \right)^p - \mu^*(V)^p \right]$$

which by iii) is

$$\leq c/2 + \sum_E \left(\int_V \mu^*(I) - \mu^*(V) \right) \leq c$$

and the conclusion follows.

Our next lemma will require the following theorem of Appling's [3].

THEOREM 2 A. *Suppose each of α and β is a function from F into the nonnegative reals, each of the $\int \alpha$ and $\int \beta$ exists, and $0 < r < 1$. Then $\int \alpha^r \beta^{1-r}$ exists, $\int \left(\int \alpha\right)^r \left(\int \beta\right)^{1-r}$ exists and they are equal.*

LEMMA 2.2. *There exists a sequence (r_n), such that $r_n \to 1$ and for each n we have $\int \mu^{r_n} = \int (\mu^*)^{r_n}$.*

Proof. For $n = 0, 1, 2, \cdots$, let $r_n = (2^n + 1)/2^n$ and notice that $r_n = 2$ if $n = 0$ and the statement holds [1]. Now suppose the conclusion holds for r_n. Notice that if $0 < x, 0 < y$ and $x + y = r_n$, then $(\mu^*)^{x+y} \leq (\mu^*)^x \mu^y \leq \leq \mu^{x+y}$. Therefore $\int (\mu^*)^x \mu^y$ exists and is $\int (\mu^*)^{x+y}$. Now $r_{n+1} = (1/2)(1 + r_n)$ so that

$$\int \mu^{r_{n+1}} = \int (\mu\mu^{r_n})^{\frac{1}{2}} = \int (\mu \int \mu^{r_n})^{\frac{1}{2}} = \int [\mu \int (\mu^*)^{r_n}]^{\frac{1}{2}} = \int |[\mu^* \mu (\mu^*)^{(\frac{1}{2})^n}]^{\frac{1}{2}} =$$

$$= \int [(\int \mu^*) \int (\mu (\mu^*)^{(\frac{1}{2})^n})]^{\frac{1}{2}} = \int [\int \mu^* \int (\mu^*)^{r_n}]^{\frac{1}{2}} = \int [\mu^* (\mu^*)^{r_n}]^{\frac{1}{2}} = \int (\mu^*)^{r_{n+1}}.$$

Before proving the main theorem we note that conventional manipulations of iii yield the following. If $1 < q < p$, and $0 < a < b < 1/e$, then

$$b^p - a^p < b^q - a^q .$$

THEOREM 2.1. *If* $p > 1$, *then* $\int \mu^p = \int (\mu^*)^p$.

Proof. Let $c > 0$, n be such that $q = (1 + 2^n)/2^n < p$ and $D \ll \{S\}$ such that if $E \ll D$, then

$$\left| \int_S \alpha(I) - \sum_E \alpha(I) \right| < c$$

for each α in $\{\mu^p, \mu^q, (\mu^*)^p, (\mu^*)^q\}$.

Now

$$0 \leq \int_S \mu(I)^p - \int_S (\mu^*(I))^p \leq 2c + \sum_D \mu(I)^p - \sum_D (\mu^*(I))^p =$$

$$= 2c + \sum_D (\mu(I)^p - (\mu^*(I))^p) \leq 2c + \sum_D (\mu(I)^q - (\mu^*(I))^q) \leq$$

$$\leq 4c + \int_S \mu(I)^q - \int_S (\mu^*(I))^q = 4c.$$

3. p ORDER LIPSCHITZ CLASSES

We now consider a Lipschitz condition for elements of $ba(F)$, the set of bounded real valued functions on F which are additive on disjoint elements of F.

DEFINITION. If $p > 0$ and $\eta \in ba(F)$, then $\eta \in \text{Lip}(\mu, p)$ iff there exists a $K > 0$ such that $|\eta(V)| \leq K\mu(V)^p$ for each $V \in F$.

For $p = 1$ we have the Lip (μ) of [2].

If we consider the set $[a, b)$ with the field of finite unions of half open intervals and require that the inequality hold only for intervals we obtain an analogue of the set of real valued functions defined on $[a, b]$ which satisfy a uniform Lipschitz condition of order p with respect to an increasing function

m [4]. For an arbitrary metric space the set of functions which satisfy a p order Lipschitz condition with respect to the metric has been studied in [5] and [6].

If $p > 1$, η is in $ba(F)$, $K > 0$ and $|\eta(I)| \leq K\mu(I)^p$ for each $I \in F$, then $\int_V |\eta(I)| \leq K \int_V \mu(I)^p$ for each $V \in F$. Furthermore $|\eta(V)| \leq \int_V |\eta(I)|$ and $\int_V |\eta(I)^p| \leq \mu(V)^p$ (ii) for each $V \in F$. Hence, $\eta \in \mathrm{Lip}(\mu, p)$ iff there exists a $K > 0$ such that $\int |\eta| \leq K \int \mu^p$ and therefore we have the following consequence of Theorem 2.1.

COROLLARY. *If* $p > 1$, *then* $\mathrm{Lip}(\mu, p) = \mathrm{Lip}\left(\int \mu^*, p\right)$.

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