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A correspondence between a class of near—rings and a class of groups

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Algebra. — *A correspondence between a class of near-rings and a class of groups.* Nota di MIRELA ȘTEFĂNESCU, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si studia una corrispondenza di Mal'cev fra classi speciali di quasi-anelli (near-rings) e di gruppi. Si mostra il carattere funtoriale di tale corrispondenza e si generalizzano alcuni risultati di Mal'cev e Weston riguardanti l'equivalenza sintattica fra le teorie di alcune classi di anelli e gruppi.

A. I. Mal'cev [4] constructed a correspondence between the class of nonassociative rings with identity and an axiomatizable class of nilpotent groups. This correspondence is an equivalence between their formalized theories (in the sense of [6]). K. Weston [7] generalized Mal'cev's result for the the class of all nonassociative rings and a larger class of nilpotent groups.

The aim of our paper is to show that there exists a correspondence of the same type between a class of distributive nonassociative near-rings and a class of groups, such that the Mal'cev's and Weston's correspondences are obtained as its restrictions. Some remarks concerning the functorial aspect of this correspondence are made.

For topics concerning near-rings, we send to [1] or [2]. We give firstly a few necessary definitions.

DEFINITION 1. A *distributive nonassociative near-ring* is a triple $(N, +, \cdot)$, where N is a nonvoid set, $+$ and $\cdot$ are binary operations on N, N is a group with respect to $+$ and $\cdot$ is left and right distributive with respect to $+$.

If the near-ring has also the property:

$$x \cdot y + z = z + x \cdot y, \qquad \forall x, y, z \in N, \tag{1}$$

then we call it a *centrally-additive distributive nonassociative near-ring* and we denote the class of those near-rings by $\mathscr{D}$.

This notion is a slight generalization of that of ring, but there exist such near-rings which are not rings. For instance, let $(N, +)$ be a metabelian group. Defining the multiplication on it by: $x \cdot y = [x, y] = -x - y + x + y$, $\forall x, y \in N$ (the commutator of x, y), then $(N, +, \cdot)$ is a member of $\mathscr{D}$ (it is associative). Let $(R, +, \cdot)$ be a nonassociative ring, $(G, +)$—a nonabelian group. The $N = R \times G$ is a member of $\mathscr{D}$, with respect to the operations: $x + y = (x_1 + y_1, x_2 + y_2)$, $x \cdot y = (x_1 \cdot y_1, 0)$, $\forall x = (x_1, x_2)$, $y = (y_1, y_2) \in N$.

We denote: by $\mathscr{D}_1$—the subclass of $\mathscr{D}$ which contains those near-rings N with $(N, +)$ a metabelian groups, by $\mathscr{D}_2$—the subclass of $\mathscr{D}_1$ formed by

(*) Nella seduta del 16 aprile 1977.

all nonassociative rings and by $\mathscr{D}_3$—its subclass containing all unitary nonassociative rings.

REMARK 2. The class $\mathscr{D}$ (as the object class) and the near-ring homomorphisms (as morphisms) form a category, $\tilde{\mathscr{D}}$, with the category of nonassociative rings, $\tilde{\mathscr{D}}_2$ (as well as $\tilde{\mathscr{D}}_1$) as its full subcategory.

DEFINITION 3. We denote by $\mathscr{G}$ *the class of those groups* G which satisfy the following conditions:

(i) There are two endomorphisms of G, α and β, such that $\alpha \circ \alpha = \beta \circ \beta = \alpha \circ \beta = \beta \circ \alpha = \theta$ (the null endomorphism of G);

(ii) Denote $H = \operatorname{Ker} \alpha \cap \operatorname{Ker} \beta$. There exist two group homomorphisms $\tilde{\alpha} : H \to \operatorname{Ker} \beta$, $\tilde{\beta} : H \to \operatorname{Ker} \alpha$, such that $(\alpha \circ \tilde{\alpha})(x) = (\beta \circ \tilde{\beta})(x) = x$, $\forall x \in H$;

(iii) The elements of H are permutable with those of $\tilde{\alpha}(H)$ and $\tilde{\beta}(H)$.

Denote by $\mathscr{G}_1$ the subslass of $\mathscr{G}$ containing those G of $\mathscr{G}$, for which $(H, +)$ is a metabelian group, while $(G, +)$ is a nilpotent group of class at most 3; by $\mathscr{G}_2$—its subclass formed by $G \in \mathscr{G}_1$ with Ker α and Ker β abelian groups; by $\mathscr{G}_3$—the subclass of $\mathscr{G}_2$, such that, for any $G \in \mathscr{G}_3$, α and β are given by: $\alpha(x) = [a_1, x]$, $\beta(x) = [a_2, x]$, $\forall x \in G$, where a_1 and a_2 are two suitable elements of G.

We immediatly obtain:

COROLLARY 4. (i). *For every x of* H, $(\alpha \circ \tilde{\beta})(x) = (\beta \circ \tilde{\alpha})(x) = 0$. (ii) *The elements* $[\tilde{\beta}(y), \tilde{\alpha}(x)]$ *are in* H, *for every* x, y *from* H.

The first part is obvious. For the second one, we apply α and β to the both sides of the equality: $[\tilde{\beta}(y), \tilde{\alpha}(x)] = -\tilde{\beta}(y) + \tilde{\alpha}(x) + \tilde{\beta}(y) + \tilde{\alpha}(x)$.

COROLLARY 5. *The class of objects* $(G, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$, *where* $G \in \mathscr{G}$, *the mappings* $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ *were defined in Definition 3, together with the group homomorphisms* $\varphi : G \to G'$ *such that* $\alpha' \circ \varphi = \varphi \circ \alpha$, $\beta' \circ \varphi = \varphi \circ \beta$ *and* $\tilde{\alpha}' \cdot \varphi_{|H} = \varphi \circ \tilde{\alpha}$, $\tilde{\beta}' \cdot \varphi_{|H} = \varphi \circ \tilde{\beta}$, *where* $\varphi_{|H}$ *is the restriction of* φ *to* H, *forms a category,* $\tilde{\mathscr{G}}$, *with* $\tilde{\mathscr{G}}_1$ *as a full subcategory.*

PROPOSITION 6. *If* N *belongs to* $\mathscr{D}$ *(resp., to* $\mathscr{D}_1$*), then* $G = N \times N \times N$ *is a group contained in* $\mathscr{G}$ *(resp., in* $\mathscr{G}_1$*) with respect to the addition of arbitrary* $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$:

$$x + y = (y_1 + x_1, y_2 + x_2, y_3 + x_2 \cdot y_1 + x_3). \tag{2}$$

Proof. It is perfectly straightforward that $(G, +)$ is a nonabelian group, with $0 = (0, 0, 0)$ and $-x = (-x_1, -x_2, -x_3 + x_2 \cdot x_1)$. We note that for proving the associativity of the addition (2), all conditions from the Definition 1, (1) included, are necessary and sufficient. The endomorphisms α and β are given by:

$$\alpha(x) = (0, 0, x_2), \quad \beta(x) = (0, 0, x_1), \qquad \forall x = (x_1, x_2, x_3) \in G. \tag{3}$$

Direct calculations show that they satisfy Def. 3, (i), and $\text{Ker}\,\alpha = \{(x_1, 0, x_3) \mid x_1, x_3 \in N\}$, $\text{Ker}\,\beta = \{(0, x_2, x_3) \mid x_2, x_3 \in N\}$, therefore $H = \{(0, 0, x) \mid x_3 \in N\}$. The mappings:

$$(4) \qquad \tilde{\alpha}((0, 0, x_3) = (0, x_3, 0), \tilde{\beta}(0, 0, x_3)) = (x_3, 0, 0), \quad \forall (0, 0, x_3) \in H,$$

are the group homomorphisms from Def. 3, (ii). For every $y = (0, y_2, 0)$ in $\tilde{\alpha}(H)$, $z = (z_1, 0, 0)$ in $\tilde{\beta}(H)$, $x = (0, 0, x_3)$ in H, we have: $x + y = y + x$, $x + z = z + x$, hence Def. 3 (iii) is verified. Now, if $N \in \mathcal{D}_1$, then one can directly verify that $(G, +)$ is nilpotent of class 3 and $(H, +)$ is metabelian.

REMARK 7. If $N \in \mathcal{D}_2$, then $\text{Ker}\,\alpha$, $\text{Ker}\,\beta$, H are abelian groups. If $N \in \mathcal{D}_3$, then $G \in \mathcal{G}_3$ and $\alpha(x) = [a_1, x]$, $\beta(x) = [a_2, x]$, where $a_1 = (-1, 0, 0)$, $a_2 = (0, 1, 0)$, with 1 the identity of N.

PROPOSITION 8. *If* $(G, +)$ *is a group from the class* $\mathcal{G}$ (*resp.*, $\mathcal{G}_1$), *then* H, *defined by Def. 3* (ii), *is a near-ring from the class* $\mathcal{D}$ (*resp.*, $\mathcal{D}_1$) *with respect to the compositions*:

$$(5) \qquad x \oplus y = y + x \quad , \quad x \odot y = [-\tilde{\beta}(y), \tilde{\alpha}(x)], \qquad \forall x, y \in H.$$

Proof. Obviously $(H, \oplus)$ is a group. Corollary 1, (ii), implies that the multiplication $\odot$ is well-defined on H. Using only the conditions from Def. 3, we prove left and right distributivity of $\odot$ with respect to $\oplus$; for instance, for the left hand, we have: $x \odot (y \oplus z) = x \odot (z + y) = [-\tilde{\beta}(z + y), \tilde{\alpha}(x)] = \tilde{\beta}(z + y) - \tilde{\alpha}(x) - \tilde{\beta}(z + y) + \tilde{\alpha}(x) = \tilde{\beta}(z) + [-\tilde{\beta}(y), \tilde{\alpha}(x)] - \tilde{\alpha}(x) - \tilde{\beta}(z) + \tilde{\alpha}(x) = x \odot z + x \odot y = (x \odot y) \oplus (x \odot z)$, $\forall x, y, z \in H$. We have also the equality (1). It is obvious that if G belongs to $\mathcal{G}_1$, then $(H, \oplus, \odot)$ belongs to $\mathcal{D}_1$.

REMARK 9. We can easily prove that $G \in \mathcal{G}_2$ implies $(H, \oplus, \odot) \in \mathcal{D}_2$.

Consider *the standard formalized theories* $\mathcal{I}_{\mathcal{D}}$ and $\mathcal{I}_{\mathcal{G}}$, in the sense of [6], for the classes $\mathcal{D}$ and $\mathcal{G}$. We note that $\mathcal{I}_{\mathcal{D}}$ has in its list of primitive symbols $\{+, \cdot, 0\}$ and $\mathcal{I}_{\mathcal{G}} - \{+, 0, \alpha(\ ), \beta(\ ), \tilde{\alpha}(\ ), \tilde{\beta}(\ ), [\ ,\ ]\}$, for denoting: algebraic operations, neutral elements, additive operators (as unary predicates), commutator brackets. We denote by " $-x$ " the element which verifies the equalities: $x + (-x) = 0 = (-x) + x$ (in $N \in \mathcal{D}$ or in $G \in \mathcal{G}$) and by P the formula of $\mathcal{I}_{\mathcal{G}}$: " $x \in \text{Ker}\,\alpha \cap \text{Ker}\,\beta$ ". Recall that the theories of two classes *are* (*sintactic*) *equivalent*, if there exist recursive mappings between them, such that a given closed formula of one of the two theories is true for any member of its class if and only if the corresponding formula (by above mappings) is true for any member of the other class. We can state the main result of this paper:

THEOREM 10. (i) *There are two mappings* $T : \mathcal{D} \to \mathcal{G}$, $T' : \mathcal{G} \to \mathcal{D}$, *such that* $(T' \circ T)(N)$ *and* N *are isomorphic near-rings for every* N *of* $\mathcal{D}$ *and*

$(T \circ T')(G)$ *and* G *are isomorphic groups for every* G *of* $\mathcal{G}$. (ii) $\mathcal{I}_{\mathcal{D}}$ *and* $\mathcal{I}_{\mathcal{G}}$ *are sintactic equivalent.* (iii) *The categories* $\tilde{\mathcal{D}}$ *and* $\tilde{\mathcal{G}}$ *are equivalent* ([5], *Ch.* II):

Proof. (i) Define $T(N) = G$, $\forall N \in \mathcal{D}$, with G given by Propositoin 6 and $T'(G) = H$, $\forall G \in \mathcal{G}$, with $(H, \oplus, \odot)$ given by Proposition 8. The isomorphisms $\tau : N \to T'(T(N))$ and $\sigma : T(T'(G)) \to G$ are the following mappings: $\tau(x) = (0, 0, x)$, $\forall x \in N$ and $\sigma((x_1, x_2, x_3)) = \tilde{\beta}(x_1) + \tilde{\alpha}(x_2) + x_3$, $\forall (x_1, x_2, x_3) \in T(T'(G))$, thence $x_1, x_2, x_3 \in H \subseteq G$. For τ, the proof is simple. For σ, we have: $\sigma(x + y) = \sigma((y_1 \oplus x_1, y_2 \oplus x_2, y_3 \oplus (x_2 \odot y_1) \oplus x_3) = \tilde{\beta}(x_1 + y_1) + \tilde{\alpha}(x_2 + y_2) + x_3 + [-\tilde{\beta}(y_1), \tilde{\alpha}(x_2)] + y_3 = (\tilde{\beta}(x_1) + \tilde{\alpha}(x_2) + x_3) - \tilde{\alpha}(x_2) + \tilde{\beta}(y_1) + \tilde{\alpha}(x_2) + \tilde{\alpha}(y_2) + [-\tilde{\beta}(y_1), \tilde{\alpha}(x_2)] + y_3 = \sigma(x) + \sigma(y)$, $\forall x, y \in T(T'(G))$. Now σ is injective, since $\sigma((x_1, x_2, x_3)) = 0$ implies $\tilde{\beta}(x_1) + \tilde{\alpha}(x_2) + x_3 = 0$, and, consequently, applying α and β, it gives in turn $x_1 = x_2 = x_3 = 0$.

To show that σ is onto, first we note that, for any $x \in G$, $x_1 = \beta, (x)$, $x_2 = \alpha(x)$ and $x_3 = -\tilde{\beta}(x_1) + x - \tilde{\alpha}(x_2)$ belong to H, and then we have: $\sigma((x_1, x_2, x_3)) = x$ (direct calculation).

(ii) Define a recursive mapping $\overline{T} : \mathcal{I}_{\mathcal{D}} \to \mathcal{I}_{\mathcal{G}}$ in the following manner: Let A be a closed formula of $\mathcal{I}_{\mathcal{D}}$. Then $\tilde{A}$, obtained from A by replacing $x_i + x_j$ by $x_j + x_i$, 0 by 0, $x_i \cdot x_j$ by $[-\tilde{\beta}(x_j), \tilde{\alpha}(x_i)]$, is a formula of $\mathcal{I}_{\mathcal{G}}$. Now, $\overline{T}(A) = \tilde{A}^{(P)}$, where $\tilde{A}^{(P)}$ is obtained by relativizing $\tilde{A}$ to P ([6], 1.5, p. 25). By Proposition 6, we see that A is true on $N \in \mathcal{D}$ if and only if $\overline{T}(A)$ is true on $T(N) \in \mathcal{G}$.

For the converse, assume that the closed formula B of $\mathcal{I}_{\mathcal{G}}$ is transformed into its prenex form: $B = (Q_1 x_1) \cdots (Q_n x_n) B_1(x_1, \cdots, x_n, 0)$, where Q_i represents a quantifier and the formula B_1 does not contain other quantifiers (see [3], II, § 3.5). We construct $\overline{T}'(B)$ in $\mathcal{I}_{\mathcal{D}}$, by replacing $(Q_i x_i)$ by $(Q_i x_i)(Q_i y_i)(Q_i z_i)$, $i = 1, 2, \cdots, n$, and expressions of the form $x_i + x_j = x_k$ by $(x_j + x_i = x_k) \wedge (y_j + y_i = y_k) \wedge (z_j + y_i \cdot x_j + z_i = z_k)$. As it is obvious from the construction of T′, B is true on G if and only if $\overline{T}'(B)$ is true on $T'(G)$.

(iii) The representative functors defining the equivalence between $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{G}}$ are the following ones: $\tilde{T}(N) = T(N)$, $\forall N \in \mathcal{D}$, $\tilde{T}(\eta) = (\eta, \eta, \eta)$, $\forall \eta \in \mathrm{Hom}_{\tilde{\mathcal{D}}}(N, N')$ and $\tilde{T}'(G) = H$, $\forall G \in \mathcal{G}$, $\tilde{T}'(\varphi) = \varphi_{|H}$, $\forall \varphi \in \mathrm{Hom}_{\tilde{\mathcal{G}}}(G, G')$. To prove that $\tilde{T} : \tilde{\mathcal{D}} \to \tilde{\mathcal{G}}$ and $\tilde{T}' : \tilde{\mathcal{G}} \to \tilde{\mathcal{D}}$ define a natural equivalence, we use the isomorphisms τ and σ from Proposition 8.

REMARK 11. Using Propositions 6 and 8 and Remarks 7 and 9, we immediately prove that the restrictions of T and T′ at $\mathcal{D}_1$ (resp. $\mathcal{D}_2$, $\mathcal{D}_3$) and $\mathcal{G}_1$ (resp. $\mathcal{G}_2$, $\mathcal{G}_3$) are correspondences of the same type as in Theorem 10 between these classes.

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