Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On the maximal subgroups of the Mathieu groups I: M_{24}

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **62** (1977), n.4, p. 432–438.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1977_8_62_4_432_0>

Algebra. — *On the maximal subgroups of the Mathieu groups I*: M_{24}. Nota di RUDY J. LIST, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — I sottogruppi massimali del gruppo di Mathieu M_{24} furono determinati da Choi [3] nel 1967: egli raccolse una gran quantità di informazioni sul sistema di Steiner S (5 , 8 , 24). Qui si mostra che la quantità di informazioni sulla geometria di S (5 , 8 , 24) necessaria per determinare i sottogruppi massimali di M_{24} è molto minore di quella raccolta in [3].

1. INTRODUCTION AND NOTATION

In [4], [5] Choi determined the maximal subgroups of $M = M_{24}$ through an intricate analysis of the geometry of the Steiner system $S = S(5, 8, 24)$; M is the automorphism group of S [14]. The character table of M was first determined by Frobenius [9], and a copy of it can be found there or in [3]. In what follows elements of M with cycle types $2^8 1^8$, 2^{12}, $3^6 1^6$, 3^8 are referred to as 2_1, 2_2, 3_1, 3_2 respectively, $H \backslash K$ denotes an extension of H by K , $H \backslash\!\!\backslash K$ denotes a split extension of H by K , C_k denotes a cyclic group of order k, and $C_m^n \simeq C_m \backslash C_n$. Notation which is not explained follows [13]. We also make use of the fact that primitive groups of degree less than or equal to 20 have been determined [11] and that the fixed point set of a 2_1 is an $\dot{8}$ [12].

As an M-module $V = V_{24}(2)$ has an invariant subspace $\mathscr{C}$ of dimension 12. The nonzero elements of $\mathscr{C}$ consist of (i) 759 vectors with 8 nonzero coordinates, (ii) 2576 vectors with 12 nonzero coordinates, (iii) 759 vectors with 16 nonzero coordinates, and (iv) the vector with 24 nonzero coordinates. The nonzero elements of $\mathscr{C}$ correspond to subsets of $\Omega = \{1, \cdots, 24\}$ in the following way: If the $i^{\text{th}}, j^{\text{th}}, \cdots, k^{\text{th}}$ coordinates of $v \in \mathscr{C}$ are the nonzero coordinates of v, then v corresponds to $\{i, j, \cdots, k\} \subseteq \Omega$. The subsets of Ω corresponding to the elements (i) are the blocks of a Steiner system S on Ω. A subset of Ω corresponding to an element of (i), (ii), or (iii) is called an $\dot{8}$, $\dot{12}$, or a $\dot{16}$ respectively. The preceding observations were first made by Carmichael.

Define the length of a vector v in V to be the number of nonzero coordinates of v. If $\bar{x}$ is a nonidentity element of the M-module $V/\mathscr{C}$, it is easy to see that the minimal length of a vector in $\bar{x}$ is 1 , 2 , 3, or 4. and that if the minimal length of a vector in $\bar{x}$ is 1 , 2, or 3, $\bar{x}$ contains a unique vector minimal length, while if the minimal length is 4 , $\bar{x}$ contains precisely six vectors of length 4. Furthermore, if the minimal length of vectors in $\bar{x}$ is 4, the union of the sets corresponding to any two distinct vectors of minimal length is an $\dot{8}$. It follows easily that any intransitive subgroup of M is contained in a conjugate of one of the following: (i) $M_{(\Delta(i))}$, where $\Delta(i) =$

(*) Nella seduta del 16 aprile 1977.

$= \{1, \cdots, i\}$, $i \leq 4$, or (ii) $M_{(E)}$, where E is $\dot{8}$, $\dot{12}$ or the fixed point set of a 3. These observations and this method of determining the intransitive subgroups of M are due to Conway [6].

Using standard methods it is also easy to show that the only proper subgroups of M primitive on Ω are conjugates of $PSL_2(23)$. For example, it is not difficult to show, using Sylow's theorem, that a proper subgroup H of M acting primitively on Ω has the same order as the order of $PSL_2(23)$, and then an argument entirely similar to that used to prove Satz 6.15 [10] shows that H is isomorphic to $PSL_2(23)$. That M contains such an H is proved in [14] and follows from the facts: The linear transformations $\alpha : x \to x + 1$, and $\beta : x \to -x^{-1}$, $x \in F_{23}$, generate $PSL_2(23)$; α and β fix a Steiner system S constructed on the points of the projective line with 24 points. That all such H are conjugate in M is clear, for: Elements of order 23 and 11 are selfcentralizing in M. Hence the groups

$$H_{x,y,z} = \langle x , y , z : x^{23} = y^{11} = z^2 , \langle x , y \rangle \simeq C_{23}^{11} , \langle y , z \rangle \simeq C_{11}^{2} \rangle$$

form a single conjugate class in M by Sylow's theorem. By the classification of the subgroups of $PSL_2(q)$ [8], C_{23}^{11} is a maximal subgroup of $PSL_2(23)$, so $H_{x,y,z} \simeq PSL_2(23)$.

To complete a determination of the maximal subgroups of M, it is necessary only to determine the subgroups imprimitive on Ω. With the method developed by Conway and outlined above it is possible to obtain some information about the imprimitive subgroups of M [6]. Here using methods alternative to those used by Choi and Conway a determination of the imprimitive subgroups of M is given.

II.

In this section we exhibit four subgroups G_1, G_2, G_3, G_4 of M which act imprimitively on Ω with block lengths 4, 8, 12, 3 and orders $2^{10}.3^3.5$, $2^{10}.3^2.7$, $2^7.3^3.5.11$, and $2^3.3.7$ respectively.

(1) Let E be an $\dot{8}$. Then $M_{[E]} \simeq W$, an elementary abelian group of order 16 acting regularly on $\Omega - E$, and $M_{(E)} \simeq W \diagdown GL_4(2)$ [12]. Hence a Sylow 2-subgroup of M is isomorphic to a Sylow 2-subgroup of $GL_5(2)$. Examination shows that a Sylow 2-subgroup of M contains precisely two elementary abelian groups of order 64. One of these has 6 orbits of length 4 on Ω, and the other has 3 orbits of length 8. Thus if H is any elementary abelian group of order 64 in M, H is M-characteristic in any Sylow 2-subgroup containing it. It follows that two elements of H are conjugate in M if and only if they are conjugate in $N_M(H)$.

Now suppose that H is an elementary abelian subgroup of M of order 64 with 6 orbits $\Delta_1, \cdots, \Delta_6$ of length 4. Then $M_{(\Delta_i)} \simeq M_{(\Delta(4))}$, because M is 4-transitive, and $M_{(\Delta_i)} \subseteq N = N_M(H)$. Also, because M is 5-transitive, $M_{(\Delta_i)}$ is transitive on $\Omega - \Delta_i$. Hence N is an imprimitive group of block length 4. Now $|M_{[\Delta(4)]}| = 2^6.3.5$, so $|M_{(\Delta)(4)}| = 2^9.3^2.5$. Since $M_{(\Delta_i)}$ is the stabilizer of a block in N represented on the blocks $\Delta_1, \cdots, \Delta_6$, it follows that $|N| = 2^{10}.3^3.5$. It is easy to show that there are no characters corresponding to induced characters of possible proper subgroups of M containing N properly. Hence N is a maximal subgroup of M. Denote N by G_1. G_1 was first determined by Todd [12].

REMARK. It is routine to show that H contains 45 elements of type 2_1 and 18 of type 2_2. Since G_1 is transitive on 2_2-involutions, it follows that if x is 2_2 in H, then $|C_{G_1}(x)| = 2^9.3.5. = |C_M(x)|$, i.e., the centralizer of an element of type 2_2 is a subgroup of a conjugate of G_1.

(2) M is transitive on the set of $\dot{8}$'s, because they are blocks of a Steiner system S. Given an $\dot{8}$ E it is not hard to show that there are precisely 30 $\dot{8}$'s disjoint from E and that $M_{(E)}$ acts transitively on this set of 30 objects. Hence M is transitive on the 3795 ordered triples of mutually disjoint $\dot{8}$'s. If H is the stabilizer of some such ordered triple (X, Y, Z), $|H| = 2^{10}.3^2.7$. It follows that H must be transitive on Ω. Again using the characters of M it is not difficult to show that H is maximal in M. Denote H by G_2.

REMARK. Let K be the kernel of imprimitivity of H. As K is a subgroup of $M_{(E)}$, where E is some $\dot{8}$, it follows that $K \simeq V\diagdown(W\diagdown PSL_2(7))$, where V and W are both elementary abelian of order 8. Take $P \subseteq K$, with $P \simeq PSL_2(7)$. It is obvious, then, that $3^2 \mid |N_H(P)|$. Hence $3 \mid |C_H(P)|$. But if σ is an element of type 3_2, $|C_M(\sigma)| = 3.|PSL_2(7)|$. Hence $C_M(\sigma) \subseteq H$. Then since $[H : C_M(\sigma)] = 2^7$, and since the centralizer in M of elements of order 7 has order 2.3.7, it follows (by applying Sylow's theorem to H forthe prime 7) that $N_M(\langle\sigma\rangle) \subseteq H$.

(3) There are $2^4.7.23$ $\dot{12}$'s in $\mathscr{C}$, so if F is a $\dot{12}$, the stabilizer $M_{(F)}$ of F must have order at least $2^6.3^3.5.11$ and must be isomorphic to a subgroup of S_{12}. the subgroups of S_{12} are known, and it follows that $M_{(F)}$ must be isomorphic to M_{12}, the Mathieu group of degree 12, and that M is transitive on the set of $\dot{12}$'s. It is obvious that M is transitive on the 1288 unordered pairs of disjoint $\dot{12}$'s, so that $M_{(F)}$ is contained in an overgroup $\overline{M}_{(F)}$ with index 2. It follows that $\overline{M}_{(F)}$ is maximal. Moreover $\overline{M}_{(F)}$ is isomorphic to Aut (M_{12}) [14]. Denote $\overline{M}_{(F)}$ by G_3.

(4) In [5] generators are exhibited (denoted here by m and n) of a subgroup of M satisfying the relations $m^2 = n^3 = (mn)^7$. Further m is 2_2, and n is 3_2. It is straightforward to verify that $\langle m, n\rangle$ satisfies the relations required in order that $\langle m, n\rangle$ be isomorphic to $PSL_2(7)$ [7]. Since m and n act semi-regularly on Ω, it follows immediately that $\langle m, n\rangle$ is imprimitive on Ω with blocks of length 3. Computing from the character table of M we find that there are 7 solutions of the equation $x \cdot y = z$, where x and y are 2_2 and 3_2 respectively, and z is a fixed element of order 7. It follows that any subgroup $\langle x, y : x \text{ is } 2_2, y \text{ is } 3_2, x \cdot y \text{ has order } 7\rangle$ of M is conjugate to $\langle m, n\rangle$.

Obviously $\langle m, n\rangle$ cannot be contained in a conjugate of G_i, $i = 1, 2, 3$. Anticipating section III, it follows that $\langle m, n\rangle$ is maximal in M. Denote $\langle m, n\rangle$ by G_4. The existence of G_4 is first noticed in [5].

III.

In this section we show that an imprimitive subgroup of M is contained in a conjugate of one of G_i, $i = 1, \cdots, 4$.

(1) *a*) *Let* H *be an imprimitive subgroup of* M *with blocks of length* 8, *and suppose that the kernel of imprimitivity* K *is transitive on each block. Then the blocks are* $\dot{8}$'s, *and* H *is contained in a conjugate of* G_2.

b) *Let* H *be an imprimitive subgroup of* M *of block length* 12. *Then the blocks are* $\dot{1}2$'s *and* H *is contained in a conjugate of* G_3.

c) *Let* H *be a non-solvable imprimitive subgroup of* M *of block lenght* 4. *Then* H *is contained in a conjugate of* G_1 *or* G_3.

Proof. *a*) Since K has orbits of length 8, $K \subseteq M_{(E)}$ for some $\dot{8}$ E. Thus the blocks are $\dot{8}$'s, and H is contained in a conjugate of G_2.

b) In this case H has just two blocks, and so it is obvious that the kernel of imprimitivity must be transitive on each block. The rest of the proof is similar to the proof of *a*).

c) Let K be the kernel of imprimitivity. If H is non-solvable, then H/K is non-solvable, because K must be solvable. Since S_5 is the only non-solvable subgroup of S_6 with a subgroup of index 24, it follows that either $K \neq 1$ or $K = 1$ and $H = S_5$. If $H = S_5$, then an $A_5 \subseteq H$ must have orbits of length 12, so $H \subseteq G_3$.

Suppose that $K \neq 1$, and let $N \subseteq K$ be a minimal normal subgroup of H. Clearly N is elementary abelian 2-group. If the orbits of N have length 2, it follows that either (i) $N = \langle\sigma\rangle$, where σ is 2_2, or (ii) nonidentity elements of N are 2_1, and distinct nonidentity elements of N have disjoint fixed point sets (since M contains just two types of involutions). In case (i) $H \subseteq C_M(\sigma) \subseteq G$, where G is a conjugate of G_1. In case (ii) H must permute the fixed point sets of the nonidentity elements of N among themselves. Since the fixed point set of a 2_1 is an $\dot{8}$, H is contained in a conjugate of G_2. But then, since H is nonsolvable, the image of imprimitivity on orbits of N is a transitive non-solvable group of degree 8, so $7 \mid |H|$. This contradicts the assumption that H can be represented as an imprimitive group of block length 4, since the image of imprimitivity over blocks of length 4 must be a subgroup of S_6. Therefore case (ii) is impossible.

Suppose that the orbits of N have length 4, and denote them by Γ_i, $i = 1, \cdots, 6$. We want to show that these are the blocks of imprimitivity of a conjugate of G_1 (introduced in section II). This will follow if we show that $\Gamma_i \cup \Gamma_j$ is an $\dot{8}$ for any pair of distinct integers i and j, $1 \leq i < j \leq 6$, for the following reasons: Given Γ_1 and $\alpha \in \Omega - \Gamma_1$ there is a unique $\dot{8}$ b incident with $\Gamma_1 \cup \{\alpha\}$ by the definition of S (5, 8, 24), so that $b - \Gamma_1$ is a uniquely determined Γ_j. By 4-transitivity of M on Ω we may assume that Γ_1 is Δ_1 in the discussion of section II where G_1 is introduced. Since nonsolvable groups of degree 6 are 2-transitive, it suffices to show that $\Gamma_1 \cup \Gamma_j$ is an $\dot{8}$ for some Γ_j.

Assume that $\Gamma_1 \cup \Gamma_j$ is not an $\dot{8}$ for $2 \leq j \leq 6$. Let b be an $\dot{8}$ incident with the elements of Γ_1 and let m denote $\max\{|b \cap \Gamma_j|, 2 \leq j \leq 6\}$. There are three cases to consider, viz., $m = 1, 2$, or 3.

Case $m = 3$; We may assume that $|b \cap \Gamma_2| = 3$ by 2-transitivity of H on the orbits of N. But then by the transitivity of N on Γ_2, $\sigma(b \cap \Gamma_2) \neq b \cap \Gamma_2$ for some $\sigma \in N$. Hence $\sigma(b) \neq b$, while $|\sigma(b) \cap b| \geq 6$. This contradicts the fact that 5 points of Ω determine a unique $\dot{8}$. Hence case $m = 3$ is impossible.

Case $m = 2$: Assume that $|b \cap \Gamma_k| = 2$, while $|b \cap \Gamma_j| = 1$, some $j \neq k$. Again it is easy to see (because of the transitivity of N on Γ_k and Γ_j) that there must be two distinct $\dot{8}$'s which intersect in at least 5 points of Ω. Hence if $|b \cap \Gamma_k| = 2$, then $|b \cap \Gamma_j| = 2$, for some uniquely determined $j \neq k$. But then since H must preserve intersection properties of the Steiner system, and since $H_{(\Gamma_1)}$ acts transitively on $\Sigma = \{\Gamma_2, \cdots, \Gamma_6\}$, this just means that this representation of $H_{(\Gamma_1)}$ must be imprimitive of block length 2. But this is impossible, because $|\Sigma| = 5$, and $2 \nmid 5$. Hence case $m = 2$ is impossible.

Case $m = 1$: In this case $b = \Gamma_1 \cup \{\alpha, \beta, \gamma, \delta\}$ where $\alpha, \beta, \gamma, \delta$ lie in pairwise distinct orbits of N disjoint from Γ_1. Suppose these are $\Gamma_2, \cdots, \Gamma_5$. By transitivity of N on each Γ_i, it follows that each element of each Γ_i, $2 \leq i \leq 5$, is in some $\dot{8}$ containing Γ_1. Given $x \in \Gamma_6$, there is a unique $\dot{8}$ c incident with $\Gamma_1 \cup \{x\}$. Since $m = 1$, $c \cap \Gamma_j \neq \varnothing$, some j, $2 \leq j \leq 5$. But then there are again two distinct $\dot{8}$'s incident with a common 5-subset of Ω. Hence case $m = 1$ is impossible.

Thus $\Gamma_i \cup \Gamma_j$ is an $\dot{8}$, $1 \leq i < j \leq 6$, and so H is a subgroup of a conjugate of G_1.

REMARK. It is easy to show, using the fact that involutions of M are 2_1 or 2_2, that a nontrivial elementary abelian 2-group of M with no orbit of length 4 or greater has order 2 or 4.

(2) *Let* H *be a nonsolvable imprimitive subgroup of* M *of block length* 3 *with kernel of imprimitivity* K.

a) *If* $K \neq 1$, $H \subseteq N_M(\langle\sigma\rangle)$, *where* σ *is* 3_2.

b) *If* $K = 1$, H *is contained in a conjugate of* G_4.

Proof. *a*) Let $N \subseteq K$ be a minimal normal subgroup of H. Then N must be elementary abelian of order 3, since subgroups of order 9 of M have an orbit length 9 (this can be most easily seen by using the fact that M contains elements of order 3 only of types 3_1 or 3_2).

b) H must have a faithful transitive representation of degree 8. The only nonsolvable groups of degree 8 which have a subgroup of index 24 are $PSL_2(7)$, $PGL_2(7)$, and $A_3(2)$, the affine group of dimension 3 over F_2. ($A_3(2) \simeq V_3(2) \cdot GL_3(2)$).

H cannot be isomorphic to $PGL_2(7)$, because M contains two classes of elements of order 7, while $PGL_2(7)$ contains only one.

Suppose that H is isomorphic to $A_3(2)$. Then $V_3(2)$ must have orbits of length 4 or 8 on Ω, and these must be blocks for a system of imprimitivity in either case. This is the situation of III (1) *a*) or *c*). It is easy to show though, that $A_3(2)$ cannot have such representation in any case.

Hence H is isomorphic to $PSL_2(7)$. $|H_{[x]}| = 7$, where $x \in \Omega$, and so involutions and elements of order 3 in H are 2_2 and 3_2 respectively. Thus H is contained in a conjugate of G_4.

(3) *If* H *is an imprimitive nonsolvable subgroup of* M *of block length* 6, *then* H *is contained in a conjugate of* G_1 *or* G_3.

Proof. Let K be the kernel of imprimitivity of H. Then K is nonsolvable. Let $N \subseteq K$, and suppose that N is a minimal normal subgroup of H. Then N must be simple; either $N \simeq A_5$ or $N \simeq A_6$. Suppose first that $N \simeq A_6$. No involution of M is centralized by a group of order 3^2. Since $|\mathrm{Aut}(N)| = 4 \cdot |A_6|$, if $3 \mid |H/N|$, then $3 \mid |C_M(N)|$, so that an involution is N would be centralized by a group of order 3^2. Therefore either $|H| = 4 \cdot |A_6|$ or $|H| = 8 \cdot |A_6|$. In either case H/N must be represented imprimitively on the set of orbits of N, so that H may be represented as an imprimitive group of block length 12. Hence H is contained in a conjugate of G_3.

Now suppose that N is isomorphic to A_5. If H acts imprimitively on the orbits of N, it may be represented as an imprimitive group of block length 12. Thus we assume that H/N acts primitively on the orbits of N, so either $H/N \simeq A_4$ or $H/N \simeq S_4$. Since $\mathrm{Aut}(A_5) \simeq S_5$ and A_4 has no subgroup of index 2, either $H \simeq N \times B$ or $H \simeq (N \times B) \setminus C_2$, where $B \simeq A_4$. The elements of order 2 in B are 2_2, because they commute with an element of order 5. Hence a 4-group in B has orbits of length 4 on Ω and is normal in H. Therefore H is contained in a conjugate of G_1.

(4) *If* H *is a nonsolvable imprimitive subgroup of* M *of block length* 2, *then* H *is contained in one of* G_i, $i = 1, 2, 3$.

Proof. Let K be the kernel of imprimitivity of H. If $K \neq 1$, either $|K| = 2$, whence $H \subseteq C_M(\sigma)$, where σ is 2_2, and H is contained in a conjugate of G_1; or $|K| = 4$ and H is a subgroup of a conjugate of G_2 as in III (1) *c*). Thus we may assume that $K = 1$. If the representation of H on the set of blocks is imprimitive, H may be represented on Ω as an imprimitive group of block length 4, 8, 12, or 6. As H is nonsolvable, it is easy to see that either III (1) *a*), *b*), *c*) or III (3) implies. Thus we may assume that H is primitive on the set of blocks. The only primitive group of degree 12 which has a subgroup of index 24 is $PGL_2(11)$ [11]. If H is isomorphic to $PGL_2(11)$, a $PSL_2(11)$ in H must have two orbits of length 12, so H is a subgroup of a conjugate of G_3.

(5) *No solvable group is a maximal subgroup of* M.

Proof. Suppose that H is a solvable maximal subgroup of M. No maximal intransitive subgroup is solvable. Thus H must be imprimitive, since 24 is not a prime power. A minimal normal subgroup N is elementary abelian, and the orbits of N are blocks of imprimitivity. Hence N is either a 2-group or a 3-group. If N is a 3-group, $N = \langle\sigma\rangle$, where σ is 3_2. In this case H is contained in a conjugate of G_2.

If N is a 2-group, N has orbits of length either (i) 2, (ii) 4, or (iii) 8.

(i) If N has 12 orbits of length 2, either $N = \langle\sigma\rangle$, where σ is 2_2, or N is a 4-group, nonidentity elements of N are 2_1, and distinct nonidentity elements have disjoint fixed point sets. As in the proof of III (1) *c*) H must be contained in a conjugate of G_1 or G_2.

(ii) If N has orbits of length 4, the image of imprimitivity must be imprimitive on the set of 6 blocks, because primitive groups of degree 6 are nonsolvable. But then H can be represented as an imprimitive group of block length 8 in which the kernel of imprimitivity is transitive on each block, or as imprimitive group of block length 12. Thus III (1) *a*) or III (1) *b*) applies.

(iii) III (1) *a*) applies.

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