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On the Multipliers of Certain spaces of Functions on the Sphere

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RENDICONTI

DELLE SEDUTE

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Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente della Classe BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On the Multipliers of Certain spaces of Functions on the Sphere.* Nota di OLUSOLA AKINYELE, presentata (*) dal Socio G. ZAPPA.

RIASSUNTO. — Si dà una caratterizzazione dei moltiplicatori entro la classe degli operatori di spazi funzionali sulla sfera a k-dimensioni ($k \geq 2$).

1. INTRODUCTION

Let S be the K-dimensional sphere ($K \geq 2$). S admits a group of rotations, namely the orthogonal group SO_{K+1}, which we denote by G. Denote by $x\alpha$ the result of the action of the rotation α on the point $x \in S$. The rotation operator R_α acting on a function f and a measure μ defined on S are defined as follows:

$$R_\alpha f(x) = f(x\alpha) \qquad \text{for all} \quad x \in S\,; R_\alpha \mu(E) = \mu(E\alpha) \qquad \text{for all}$$

μ-measurable subsets $E \subset S$.

Let E and F be Banach spaces of functions on S, and let $B_G(E, F)$ denote the space of bounded linear operators of E into F which commute with the rotation operators R_α, $\alpha \in G$. In [2] Dunkl showed that if T belongs to any of the spaces $B_G(L^1(S), L^1(S))$ and $B_G(C(S), C(S))$, then there exists a unique zonal measure μ, that is, $\mu \in M(S\,; p)$ such that $Tf = f * \mu$ for f in any of the spaces $L^1(S)$ and $C(S)$. Moreover the correspondence between T and μ is an isometric isomorphism of $B_G(L^1(S), L^1(S))$ and $B_G(C(S))$ onto the Banach algebra of zonal measures denoted by $M(S\,; p)$, where p is the north pole of the sphere and $M(S\,; p) = \{\mu \in M(S) : R_\alpha \mu = \mu \text{ for all } \alpha \ni p\alpha = p\}$.

(*) Nella seduta del 12 febbraio 1977.

In this paper, we characterize the bounded linear operators which belong to any of the following spaces: $B_G(L^1(S), M(S))$, $B_G(L^1(S), L^q(S))$ $1 < q < \infty$, $B_G(L^q(S), L^\infty(S))$ $1 \leq q < \infty$ and $B_G(L^q(S), C(S))$ $1 \leq q \leq \infty$. We show in section 3, that $T \in B_G(L^1(S), M(S))$ if and only if there exists a unique $\mu \in M(S; p)$ such that $Tf = f * \mu$ for all $f \in L^1(S)$ and that $T \in B_G(L^1(S), L^q(S))$ if and only if there exists a unique $g \in L^q(S; p)$ such that $Tf = f * g$ for all $f \in L^1(S)$. We also show that $T \in B_G(L^q(S), L^\infty(S))$ if and only if there exists a unique $g \in L^r(S; p)$, $1/q + 1/r = 1$ such that $Tf = f * g$ for all $f \in L^q(S)$. Finally we show that $T \in B_G(L^q(S), C(S))$ if and only if there exists a unique $g \in L^r(S; p)$ $(1/q + 1/r = 1)$ such that $Tf = f * g$ for all $f \in L^q(S)$.

The author would like to thank Prof. C. F. Dunkl for his comments on the manuscript.

2. Preliminaries

Considering S imbedded as the unit sphere in R^{K+1} and for $x, y \in S$, let $x \cdot y$ denote the ordinary inner product of the vectors which correspond to the point x and y $(-1 \leq x \cdot y \leq 1)$. S has a unique rotation invariant Borel measure m such that $m(S) = 1$. Define $L^q(S) = L^q(S; m)$ and $M(S)$ to be the space of regular Borel measure on S. If p is the north pole of S define the zonal functions as the subspace, $L^q(S; p) = \{f \in L^q(S) : R_\alpha f = f$ for all $\alpha \ni p\alpha = p\}$. For any $x \in S$, let G_x be the closed subgroup of G leaving x fixed. It is known [2] that there is a homeomorphism between the space $S|_{G_p}$ and the closed interval $[-1, 1]$ induced by the continuous map $x \to p \cdot x$. Thus there exists an isomorphism between the zonal functions and integrable functions defined on $[-1, 1]$. With respect to this isomorphism the space of zonal functions maps isometrically onto $L^1([-1, 1], \Omega_{(K-1)/2})$ where Ω_λ is the measure $a_\lambda (1 - t^2)^{\lambda - 1/2}\, dt$ and a_λ is such that $\int_{-1}^{1} d\Omega_\lambda(t) = 1$. The polynomials $P_n^\lambda(t)$ defined by the relation $(1 - 2rt + r^2)^{-\lambda} = \sum_{n=0}^{\infty} \frac{\Gamma(2\lambda + n)}{n!\,\Gamma(2\lambda)} \cdot P_n^\lambda(t) r^n$ form a complete orthogonal set on $[-1, 1]$ with respect to the measure Ω_λ. It follows then that the functions $P_n^{(K-1)/2}(p \cdot x)$ form a complete orthogonal set of zonal functions. Throughout this paper, we shall assume that $\lambda = (K - 1)/2$, For other background material and notations we shall lean very much on [2]. For any $x \in S$, if we define $M(S; x) = \{\mu \in M(S) : R_\alpha \mu = \mu$ for all $\alpha \in G_x\}$ then the mapping $\phi_x : M(S; p) \to M(S; x)$ defined by $\phi_\alpha \mu = R_x \mu$ is an isometric isomorphism onto [cf.: 2]$\cdot$ ϕ_x can be defined similarly for all the subspaces of $M(S; p)$ which are Banach algebras.

Suppose $\mu \in M(S; p)$, $n = 0, 1, 2, \cdots$, then we define the nth Gegenbauer coefficient $\hat{\mu}_n$ of μ by

$$\hat{\mu}_n = \int_S P_n^\lambda(p \cdot x)\, d\mu(x)$$

and for $f \in L^1(S;p)$ define

$$\hat{f}_n = \int_S P_n^\lambda(p \cdot x) f(x) \, dm(x),$$

$|\hat{\mu}_n| \leq \|\mu\|$ and $|\hat{f}_n| \leq \|f\|$. For $f \in L^1(S)$ and $n = 0, 1, 2, \cdots$ we define

$$\tilde{f}_n(x) = \int_S f(y) P_n^\lambda(x \cdot y) \, dm(y).$$

Similarly for $\mu \in M(S)$ we define

$$\tilde{\mu}_n(x) = \int_S P_n^\lambda(x \cdot y) \, d\mu(y).$$

Clearly $\hat{\mu}_n = \tilde{\mu}_n(p)$ and $\hat{f}_n = \tilde{f}_n(p)$.

3. MULTIPLIERS OF CERTAIN SUBALGEBRAS OF M (S)

3.1. DEFINITION. Let T be a linear operator defined on any of the spaces $L^q(S)$ $(1 \leq q \leq \infty)$, with its range a subspace of $M(S)$. Denote by $N_0 = \{0, 1, 2, 3, \cdots\}$. If there exists a bounded complex valued function ϕ defined on $N_0 = \{0, 1, 2, 3 \cdots\}$ such that $(Tf)_n^{\sim}(x) = \phi(n) \tilde{f}_n(x)$ for all f and $x \in S$, then we call T a multiplier.

3.2. THEOREM. *Let* $T: L^1(S) \rightarrow M(S)$ *be a bounded linear operator. Then* T *is a multiplier if and only if there exists a unique* $\mu \in M(S;p)$ *such that* $Tf = \mu * f$ *for all* $f \in L^1(S)$. *Moreover* $\|T\| = \|\mu\|$.

Proof. Suppose $\mu \in M(S;p)$ and $f \in L^1(S)$, then if $Tf = \mu * f$, $Tf \in M(S)$ [2] and $\|Tf\| \leq \|\mu\| \|f\|$. Hence $(Tf)_n^{\sim}(x) = (\mu * f)_n^{\sim}(x) = \hat{\mu}_n \tilde{f}_n(x)$, $n \in \{0, 1, 2, \cdots\}$. If we define ϕ on $N_0 = \{0, 1, 2, \cdots\}$ by setting $\phi(n) = \hat{\mu}_n$, $n \in N_0$, then ϕ is a bounded complex valued function on N_0 such that $(Tf)_n^{\sim}(x) = \phi(n) \tilde{f}_n(x)$ for all $f \in L^1(S)$, so that T is a multiplier.

Conversely suppose T is a multiplier, then $\exists$ a bounded function ϕ on N_0 such that $(Tf)_n^{\sim}(x) = \phi(n) \tilde{f}_n(x)$, $x \in S$. Let $\alpha \in G$, $f \in L^1(S)$ and $x \in S$, then

$$(R_\alpha f)_n^{\sim}(x) = \int_S R_\alpha f(y) P_n^\lambda(x \cdot y) \, dm(y)$$

$$= \int_S f(y) P_n^\lambda(x \cdot y\alpha^{-1}) \, dm(y) = f * \psi_p^{-1} P_n^\lambda(x\alpha) = \tilde{f}_n(x\alpha) = R_\alpha(\tilde{f}_n)(x)$$

where $\psi_p^{-1} P_n^\lambda(x) \equiv P_n^\lambda(p \cdot x)$.

Define Ef to be a unique measure in $M(S)$ such that

$$(Ef)_n^{\sim}(x) = \phi(n) \tilde{f}_n(x) \quad n \in N_0, \quad x \in S.$$

Then clearly E is a bounded linear operator of $L^1(S)$ into $M(S)$. Assume that

$f \in C(S)$, then clearly Ef belongs to $C(S)$, and since $C(S)$ is dense in $L^1(S)$, the range of E is $L^1(S)$. Moreover,

$$(R_\alpha Ef)_n^\sim(x) = (Ef)_n^\sim(x\alpha) = \phi(n)\tilde{f}_n(x\alpha) = \phi(n)(R_\alpha \tilde{f}_n)(x) = \\ = \phi(n)(R_\alpha f)_n^\sim(x) = (ER_\alpha f)_n^\sim(x).$$

Therefore E is a bounded linear operator of $L^1(S)$ into itself which commutes with all rotations. Theorem 10 C of [2] implies the existence of a unique $\mu \in M(S;p)$ such that $Ef = f * \mu$ for all $f \in L^1(S)$, moreover, $\|E\| = \|\mu\|$. However, $(Ef)_n^\sim(x) = \phi(n)\tilde{f}_n(x) = (Tf)_n^\sim(x)$ for each $n \in N_0, x \in S$, hence $T = E$ and $Tf = f * \mu$.

In the next theorem we wish to obtain a characterization of the bounded linear operators in $B_G(L^1(S), L^q(S))$.

3.3. THEOREM. *Let* $T: L^1(S) \to L^q(S)$ $1 < q < \infty$ *be a bounded linear operator. Then the following statements are equivalent.*

(i) $TR_\alpha = R_\alpha T$ *for all* $\alpha \in G$,

(ii) *There exists a unique bounded continuous function* ϕ *on* $N_0 = \{0, 1, 2, 3, \cdots\}$ *such that* $(Tf)_n^\sim(x) = \phi(n)\tilde{f}_n(x)$ *for all* $f \in L^1(S)$, $n \in N_0$,

(iii) *There exists a unique function* $g \in L^q(S;p)$ *such that* $Tf = g * f$ *for all* $f \in L^1(S)$.

Moreover $\|T\| = \|g\|_q$.

Proof. Let $T: L^1(S) \to L^q(S)$ be a bounded linear operator and assume that (i) holds. Then for $f \in L^1(S;p)$ and $\alpha \in G_p$, $R_\alpha Tf = TR_\alpha f = Tf$, so that $Tf \in L^q(S;p)$. Let $h \in L^r(S)$ where $1/q + 1/r = 1$ and define a mapping

$$F: L^1(S) \to C$$

by setting

$$F(f) = \int_S Tf(x)\, h(x)\, dm(x),$$

for $f \in L_1(S)$. Clearly F is a bounded linear functional on $L^1(S)$. Thus $\exists$ a $\beta \in L^\infty(S) \ni \int_S Tf(x)\, h(x)\, dm(x) = \int_S f(x)\, \beta(x)\, dm(x)$. Let $f \in L^1(S)$ and, $g \in L^1(S;p)$, since $TR_\alpha = R_\alpha T$ for all $\alpha \in G$, the same technique of Theorem 10 a of [2] yields the following:

$$\int_S (f * Tg)(x)\, h(x)\, dm(x) = \int_S T(f * g)(x)\, h(x)\, dm(x).$$

Since this holds for all $h \in L^r(S)$, $T(f * g) = f * Tg$ for $f \in L^1(S)$, $g \in L^1(S;p)$. In particular for $f, g \in L^1(S;p)$, $T(f * g) = g * Tf = Tf * g$, and so $Tf * g = f * Tg$. Taking Gegenbauer coefficients we have $(Tf)_n^\wedge \hat{g}_n = \hat{f}_n (Tg)_n^\wedge$. Choose $f \in L^1(S;p) \ni \hat{f}_n \neq 0$ and define ϕ on $N_0 = \{0, 1, 2, 3, \cdots\}$ by setting

$\phi(n) = (\mathrm{T}f)_n^{\wedge}/\hat{f}_n$, then ϕ is a bounded continuous complex valued function independent of the choice of f and moreover $(\mathrm{T}f)_n^{\wedge} = \phi(n)\hat{f}_n$, $x \in \mathrm{S}$, $n \in \mathrm{N}_0$ for all $f \in \mathrm{L}^1(\mathrm{S}\,;p)$. Suppose $f \in \mathrm{L}^1(\mathrm{S})$ then there exists a net $\{\mathrm{U}_v\}$ of zonal functions [2, 4 *b*] such that $f * \mathrm{U}_v \to f$ (L^1 — norm).

$$[\mathrm{T}(f * \mathrm{U}_v)]_n^{\sim}(x) = (f * \mathrm{TU}_v)_n^{\sim}(x) = \tilde{f}_n(x)\,(\mathrm{TU}_v)_n^{\sim}(x) =$$
$$= \tilde{f}_n(x)\,\phi(n)\,(\mathrm{U}_v)_n^{\wedge} \to \phi(n)\,\tilde{f}_n(x)\,.$$

$[\mathrm{T}(f * \mathrm{U}_v)]_n^{\sim}(x) \to (\mathrm{T}f)_n^{\sim}(x)$ hence $(\mathrm{T}f)_n^{\sim}(x) = \phi(x)\,\tilde{f}_n(x)$ for all $f \in \mathrm{L}^1(\mathrm{S})$, $n \in \mathrm{N}_0$ and $x \in \mathrm{S}$, which implies (ii).

Now assume (ii), and let $\{\mathrm{U}_v\}$ be an approximate identity such that for any $f \in \mathrm{L}^1(\mathrm{S}\,;p)$, $f * \mathrm{U}_v \to f$ in the L^1–norm and $\|\mathrm{U}_v\| = 1$. Then for $f \in \mathrm{L}^1(\mathrm{S}\,;p)$

$$\|\mathrm{T}f - (\mathrm{TU}_v) * f\|_q = \|\mathrm{T}f - \mathrm{T}(\mathrm{U}_v * f)\|_q \leq \|\mathrm{T}\|\,\|f - \mathrm{U}_v * f\|_1 \to 0$$

Thus $\mathrm{T}f = \lim\limits_{v} \mathrm{TU}_v * f \in \mathrm{L}_q(\mathrm{S})$. For $\alpha \in \mathrm{G}$,

$$(\mathrm{R}_\alpha \mathrm{T}f)_n^{\sim}(x) = (\mathrm{T}f)_n^{\sim}(x\alpha) = \phi(n)\,\tilde{f}_n(x\alpha) = \phi(n)\,\mathrm{R}_\alpha \tilde{f}_n(x) =$$
$$= \phi(n)\,(\mathrm{R}_\alpha f)_u^{\sim}(x) = (\mathrm{TR}_\alpha f)_n^{\sim}(x)\,.$$

Hence $\mathrm{TR}_\alpha = \mathrm{R}_\alpha \mathrm{T}$ and so for $f \in \mathrm{L}^1(\mathrm{S}\,;p)$, $\mathrm{T}f \in \mathrm{L}^q(\mathrm{S})$ and $\mathrm{R}_\alpha(\mathrm{T}f) = (\mathrm{R}_\alpha \mathrm{T})f = \mathrm{TR}_\alpha f = \mathrm{T}f$. Thus $\mathrm{T}f \in \mathrm{L}^q(\mathrm{S}\,;p)$ and moreover $\|\mathrm{TU}_v\|_q \leq \|\mathrm{T}\|\,\|\mathrm{U}_v\|_1 = \|\mathrm{T}\|$. The set $\{\mathrm{TU}_v\}$ is bounded in the L_q-norm by $\|\mathrm{T}\|$ and since $\mathrm{L}^q(\mathrm{S}\,;p) = \mathrm{L}^r(\mathrm{S}\,;p)'$, there exists a subsequence [cf. 1, V. 4. 2] TU_{v_j} converging weakly to $g \in \mathrm{L}^q(\mathrm{S}\,;p)$ and so for each $h \in \mathrm{L}^r(\mathrm{S}\,;p) \ni 1/q + 1/r = 1$, we obtain

$$\lim_j \int_{\mathrm{S}} \mathrm{TU}_{v_j}(x)\,h(x)\,dm(x) = \int_{\mathrm{S}} g(x)\,h(x)\,dm(x)\,.$$

In particular we take $h = \psi_p^{-1}\,\mathrm{P}_n^\lambda \in \mathrm{C}(\mathrm{S}\,;p) \subset \mathrm{L}^r(\mathrm{S}\,;p)$ and then

$$\lim_j \int_{\mathrm{S}} \mathrm{TU}_{v_j}(x)\,\psi_p^{-1}\,\mathrm{P}_n^\lambda(x)\,dm(x) = \int_{\mathrm{S}} g(x)\,\psi_p^{-1}\,\mathrm{P}_n^\lambda(x)\,dm(x)\,,$$

i.e.
$$\lim_j \int_{\mathrm{S}} \mathrm{TU}_{v_j}(x)\,\mathrm{P}_n^\lambda(\mathrm{P}\cdot x)\,dm(x) = \int_{\mathrm{S}} g(x)\,\mathrm{P}_n^\lambda(\mathrm{P}\cdot x)\,dm(x) = \hat{g}_n.$$

Hence $(\mathrm{TU}_{v_j})_n^{\wedge} \to \hat{g}_n$. But $(\mathrm{TU}_{v_j})_n^{\wedge} = \phi(n)\,(\mathrm{U}_{v_j})_n^{\wedge} \to \phi(n)$ since $(\mathrm{U}_{v_j})_n^{\wedge} \to 1$; and so $\phi(n) = \hat{g}_n$ for all $n \in \mathrm{N}_0$. Finally for $f \in \mathrm{L}^1(\mathrm{S})$

$$(\mathrm{T}f)_n^{\sim}(x) = \phi(n)\,\tilde{f}_n(x) = \hat{g}_n\,\tilde{f}_n(x) = (g * f)_n^{\sim}(x)\,,$$

[2, 4 *a*]. Clearly $\mathrm{T}f = g * f$ for all $f \in \mathrm{L}^1(\mathrm{S})$ which is (iii). Assume that (iii) holds, then for $f \in \mathrm{L}^1(\mathrm{S})$, and $x \in \mathrm{S}$,

$$\mathrm{R}_\alpha \mathrm{T}f(x) = \mathrm{T}f(x\alpha) = g * f(x\alpha) = \mathrm{R}_\alpha(f * g)(x) = (\mathrm{R}_\alpha f * g)(x) = (\mathrm{TR}_\alpha f)(x)$$

for all $\alpha \in G$. Hence $R_\alpha Tf = TR_\alpha f$ for all $f \in L^1(S)$ and (i) holds. From (ii) $Tf = g * f$ for $g \in L^q(S; p)$ and all $f \in L^1(S)$, so $\| Tf \|_q = \| g * f \|_q \leq \| g \|_q \| f \|_1$ which implies $\| T \| \leq \| g \|_q$. Also given $\varepsilon > 0$ $\exists f \in L^1(S) \ni \| g * f \|_q > \| g \|_q - \varepsilon$ for $q < \infty$ and this implies $\| g \|_q \leq \| T \|$. Therefore $\| T \| = \| g \|_q$.

3.4. THEOREM. *Let* $T : L^q(S) \rightarrow L^\infty(S)$ *be a bounded linear operator* $1 \leq q < \infty$. *Then the following are equivalent.*

(i) T *commutes with all rotation operators,*

(ii) *There exists a unique* $g \in L^r(S; p)$, $1/q + 1/r = 1$ *such that* $Tf = g * f$ *for each* $f \in L^q(S)$, *and* $\| g \|_r = \| T \|$.

For $q = \infty$, *Theorem 3.4 remains an open problem.*

Proof. Suppose (i) holds and define T* the adjoint of T on $L^\infty(S)^*$ into $L^q(S)^*$. Then T* is a mapping of $L^1(S)$ into $L^r(S)$ where $1/q + 1/r = 1$. Now for $f \in L^q(S)$ and $h \in L^1(S)$,

$$\int_S Tf(x)\, h(x)\, dm(x) = \int_S f(x)\, T^* h(x)\, dm(x)$$

so that for $\alpha \in G$,

$$\int_S f(x)\, T^* R_\alpha h(x)\, dm(x) = \int_S Tf(x)\, h(x\alpha)\, dm(x) = \int_S Tf(x\alpha^{-1})\, h(x)\, dm(x)$$

$$= \int_S R_{\alpha^{-1}} Tf(x)\, h(x)\, dm(x) = \int_S TR_{\alpha^{-1}} f(x)\, dm(x)$$

$$= \int_S R_{\alpha^{-1}} f(x)\, T^* h(x)\, dm(x) = \int_S f(x)\, R_\alpha T^* h(x)\, dm(x).$$

Since this holds for all $f \in L^q(S)$, then $T^* R_\alpha h = R_\alpha T^* h$ for all $h \in L^1(S)$ and so T* is a bounded linear operator of $L^1(S)$ into $L^r(S)$ which commutes with all rotation operators. Therefore by Theorem 3.3 $\exists$ a unique $g \in L^r(S; p)$ such that $T^* h = g * h$ for all $h \in L^1(S)$ and $\| T^* \| = \| g \|_r$. So finally for $h \in L^1(S)$, $f \in L^q(S)$, by Fubini's theorem,

$$\int_S Tf(x)\, h(x)\, dm(x) = \int_S f(x)\, T^* h(x)\, dm(x) = \int_S f(x)\, (g * f)(x)\, dm(x)$$

$$= \int_S f(x) \int_S \{\phi_x\, g(y)\, h(y)\, dm(y)\}\, dm(x)$$

$$= \int_S f(x) \left\{ \int_S \phi_y \, g(x) \, h(y) \, dm(y) \right\} dm(x)$$

$$= \int_S \left\{ \int_S f(x) \, \phi_y \, g(x) \, dm(x) \right\} h(y) \, dm(y)$$

$$= \int_S (f * g)(y) \, h(y) \, dm(y) .$$

Since this holds for all $f \in L^q(S)$, again $Tf = f * g$, for all $f \in L^q(S)$ which is (ii). Clearly it is easy to show that $\| g \|_r = \| T \|$. Using the same technique as in Theorem 3.3 (ii) implies (i).

3.5. THEOREM. *Let* $T : L^q(S) \to C(S)$ $1 \leq q \leq \infty$ *be a bounded linear operator.* T *commutes with the rotation operators if and only if there exists a unique function* $h \in L^r(S; p)$, $1/q + 1/r = 1$ *such that* $Tf = h * f$ *for every* $f \in L^q(S)$.

Proof. Suppose $Tf = h * f$, then $\| Tf \|_\infty \leq \| h \| \, \| f \| < \infty$, and also for $x \in S$, $\alpha \in G$,

$$R_\alpha Tf(x) = Tf(x\alpha) = (h * f)(x\alpha) = R_\alpha (h * f)(x) = (h * R_\alpha f)(x) = TR_\alpha f(x).$$

So T commutes with all rotations. Now suppose $T : L^q(S) \to C(S)$ is a bounded linear operator commuting with rotations, then for $f \in L^q(S; p)$, $Tf \in C(S)$ and $R_\alpha (Tf) = TR_\alpha f = Tf$, so $Tf \in C(S; p)$. Consider the map $F : L^q(S; p) \to C$ given by

$$F(f) = Tf(p)$$

where $p = (1, 0, 0, \cdots, 0)$ the north pole of S. Since T is continuous F is a continuous linear functional on $L^q(S; p)$ and so $\exists$ a unique $h \in L^r(S; p)$, $1/q + 1/r = 1$ such that

$$F(f) = \int_S f(x) \, h(x) \, dm(x) , \qquad f \in L^q(S; p) .$$

So $Tf(p) = \int_S f(x) \, h(x) \, dm(x)$ for $f \in L^q(S; p)$. Let $\beta \in G$ such that $\beta p = p$, then for $h \in L^r(S; p)$, $R_\beta h = \phi_p h \in L^r(S; p)$ and so $\phi_p h(x) = h(x)$ for every $x \in S$. Consequently,

$$Tf(p) = \int_S f(x) \, h(x) \, dm(x) = \int_S f(x) \, \phi_p h(x) \, dm(x) = (h * f)(p) .$$

For any $x \in S$, let $\alpha \in G$ be such that $\alpha x = p$, then

$$Tf(x) = Tf(\alpha^{-1} p) = R_{\alpha^{-1}} Tf(p) = TR_{\alpha^{-1}} f(p) = h * R_{\alpha^{-1}} f(p)$$
$$= R_{\alpha^{-1}} (h * f)(p) = h * f(\alpha^{-1} p) = (h * f)(x) .$$

So for every $f \in L^q(S ; p)$, $Tf = h * f$ where $h \in L^r(S ; p)$. Now if $f \in L^q(S)$, then there exists an approximate identity $U_v \in L^1(S ; p) \cap L^q(S ; p)$ such that $f * U_v \to f$ (L^q–norm), and then,

$$\lim_v T(f * U_v) = \lim_v (f * TU_v) = \lim_v (f * U_v * h) = f * h$$

in L^q–norm.

But $\lim_v T(f * U_v) = Tf$, hence $Tf = f * h$ for all $f \in L^q(S)$.

3.6. COROLLARY.

(i) $B_G(L^1(S), C(S)) \simeq B_G(L^1(S), L^\infty(S)) \simeq L^\infty(S ; p)$

(ii) $B_G(L^\infty(S), C(S)) \simeq L^1(S ; p)$.

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