Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

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## On the orthogonal Brauer group

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, Vol. **62** (1977), n.3, p. 341–347.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1977_8_62_3_341_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1977.

**Topologia.** — *On the orthogonal Brauer group.* Nota di ELISABETTA STRICKLAND (*), presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Si ottiene per il gruppo di Brauer ortogonale BrO (X) di un CW–complesso finito X un risultato analogo a quello ottenuto da Jean Pierre Serre per il gruppo di Brauer Br (X), individuandone in modo completo la struttura omologica. Precisamente si dimostra che BrO (X) $\cong$ $H^2(X, Z_2)$. Inoltre, questo risultato ed il Teorema di Serre, che prova l'isomorfismo di Br (X) con il sottogruppo di torsione di $H^3(X, Z)$, sono messi in relazione, rappresentando gli elementi di ordine 2 di Br (X) con elementi ortogonali di BrO (X).

## § 0. INTRODUCTION

Let X be a finite CW-complex and $F_n(X)$ the set of fibre bundles on X, with fibres in the algebra of matrices $M_n(C)$, over the complex field **C**.

Consider $F(X) = \bigcup_n F_n(X)$. This set can be made into a semigroup with respect to the tensor product. Let KP (X) be the Grothendieck group of F (X), that is to say the free group generated by F (X) modulo the relations

$$a + b = a \otimes b \qquad \forall a, b \in F(X).$$

The Brauer group Br (X) of the space X is the quotient of KP (X) modulo the elements of the form End (E), E being a vector fibre bundle on X. As by Skolem-Noether [1], the automorphisms of the algebra $M_n(C)$ are all inner automorphisms, is a standard construction to see that a classifying space for KP (X) is given by the space

$$\mathrm{BPGL}(\infty) = \varinjlim \mathrm{BPGL}(n)$$

where the limit is made with respect to the maps induced by the homomorphisms

$$\mathrm{PGL}(n) \xrightarrow{\otimes \mathrm{I}} \mathrm{PGL}(nk) \qquad k \in Z^+$$

I being the unit matrix [4].

Now consider the fibration

$$\begin{array}{c} \mathrm{B}\mathbf{C}^* \\ \downarrow \\ \mathrm{BGL}(\infty)_0 \\ \downarrow \varphi \\ \mathrm{BPGL}(\infty) \end{array}$$

(*) Partially supported by G.N.S.A.G.A. of C.N.R.
(**) Nella seduta del 12 marzo 1977.

where $BGL(\infty)_0 = \varinjlim BGL(n)$ is defined similarly to $BPGL(\infty)$. Note that by the standard construction of the localization for an H-space, it follows that $BGL(\infty)_0$ is the localization of $BGL(\infty)$, the classifying space for $\tilde{K}(X)$, ([5] 2.17).

In this way an element $x \in KP(X)$ belongs to the subgroup generated by the elements of type End (E), if and only if the corresponding element $\bar{x}$ in $[X, BPGL(\infty)]$ lifts to an element of $[X, BGL(\infty)_0]$.

$BC^*$ is the Eilenberg-MacLane space $K(Z, 2)$ and we obtain

$$[X, BGL(\infty)_0] = [\tilde{K}(X) \otimes Q] \quad , \quad [X, BPGL(\infty)] = KP(X)$$

and

$$Br(X) = KP(X)/\mathrm{Im}\, \varphi_* \tilde{K}(X) \otimes Q$$

where $\varphi_*$ is the map induced by $\varphi$ among the sets of homotopy classes $\tilde{K}(X)$ and $KP(X)$.

Jean Pierre Serre has shown ([2], IV, Theorem 1.6.) that $Br(X) \cong \mathrm{Tor}\, H^3(X, Z)$, this being the torsion subgroup of $H^3(X, Z)$: such result gives a complete determination of the Brauer group in homological terms. The object of these notes is to study the problem in the orthogonal case and to find connections between this and the general linear one. Indeed, you can give in quite a similar way to the general linear case, the definition of the orthogonal Brauer group BrO (X) of a finite CW-complex X.

This is obviously obtained considering $BO(\infty)_0$ instead of $BGL(\infty)_0$ and $BPO(\infty)$ instead of $BPGL(\infty)$.

$BPO(\infty)$ is the classifying space of the functor KPO (—), which gives the Grothendieck group associated to the semigroup of fibre bundles in matrices with an orthogonal structure and $BO(\infty)_0$ is the classifying space of $\widetilde{KO} \otimes Q$.

Using homotopy theory, we are going to prove that

*a)* $KPO(X) \cong \bigoplus_{i=1}^{\infty} H^{4i}(X, Q) \oplus H^2(X, Z_2)$,

*b)* $BrO(X) \cong H^2(X, Z_2)$.

Now, in view of the fact that, if an element of Br (X) is representable by means of an orthogonal element, then it is an element of order 2 (as a matter of fact, in this case the orthogonal structure gives $E \cong E^0$, $E^0$ being the opposite space of E, and $E \otimes E^0 \cong \mathrm{End}\, E$, so $2E = 0$ in the Brauer group), it is quite natural to investigate, on the basis of Serre's theorem and result *b)*, if an element of order 2 in Br (X) is representable by means of an element in BrO (X).

This is the main result of our work: we shall give an affermative answer to this question, as

*c)* the map

$$BrO(X) \xrightarrow{F'} Br(X)$$

induced by the forgetful because of *b*) and Serre's isomorphism, corresponds to the Bockstein homorphism from $H^2(X, Z_2)$ to $H^3(X, Z)$, relative to the sequence $Z \to Z \to Z_2$. In other words, the diagram

$$\begin{array}{ccc} \mathrm{BrO}(X) & \xrightarrow{F'} & \mathrm{Br}(X) \\ \wr\| & & \wr\| \\ H^2(X, Z_2) & \to & \mathrm{Tor}\, H^3(X, Z) \end{array}$$

is commutative.

## § 1. THE HOMOLOGICAL STRUCTURE OF THE ORTHOGONAL BRAUER GROUP BrO (X)

Let us first outline what we are going to do in this section. The idea is to compute, using Bott periodicity theorem [3], the homotopy groups of $\mathrm{BPO}(\infty)$.

PROPOSITION 1.1. *There is a one-to-one map from* $\mathrm{BrO}(X)$ *to* $H^2(X, Z_2)$.

*Proof.* Take the diagram

$$(1.1) \qquad \begin{array}{ccc} K(Z_2, 1) & \to & K(Z_2, 1) \\ \downarrow & & \downarrow \\ \mathrm{BO}(\infty)_0 & \to & U \\ \downarrow \varphi & & \downarrow \\ \mathrm{BPO}(\infty) & \xrightarrow{\psi} & K(Z, 2) \end{array}$$

where U stands for the total space of the universal fibration for $K(Z_2, 1)$, (U contractible) and $\psi$ induces the fibration $\mathrm{BO}(\infty)_0 \to \mathrm{BPO}(\infty)$. $\psi$ gives, under composition, the map

$$s: \quad \mathrm{KPO}(X) \to H^2(X, Z_2).$$

Let us detect $\mathrm{Ker}(s)$. If $x \in \mathrm{KPO}(X)$ and $s(x) = 0$, then $x$ lifts to $\mathrm{BO}(\infty)_0$, so $x \in \mathrm{Im}\, \varphi_* \widetilde{\mathrm{KO}}(X) \otimes Q$. Viceversa, if $x \in \mathrm{Im}\, \varphi_* \widetilde{\mathrm{KO}}(X) \otimes Q$, the diagram (1.1) states that $s(x) = 0$, so

$$\mathrm{Ker}(s) = \mathrm{Im}\, \varphi_* \widetilde{\mathrm{KO}}(X) \otimes Q$$

Since $\mathrm{BrO}(X) = \mathrm{KOP}(X)/\mathrm{Im}\, \varphi_* \widetilde{\mathrm{KO}}(X) \otimes Q$, we are done.

In the following, the main tool is going to be forwarded by Bott's periodicity theorem [3] in the orthogonal form. Such theorem states that

$$\pi_n(\mathrm{BO}(\infty)_0) = \begin{cases} 0 & \text{if } n \neq 4i \\ Q & \text{if } n = 4i \end{cases}$$

where $\pi_n$ is the $n$ th-homotopy group.

We can immediately make use of this result, considering the exact homotopy sequence corresponding to the fibration

$$
\begin{array}{c}
K(Z_2, 1) \\
\downarrow \\
BO(\infty)_0 \\
\downarrow \\
BPO(\infty) .
\end{array}
\tag{*}
$$

We obatain

$$\cdots \to \pi_n(K(Z_2, 1)) \to \pi_n(BO(\infty)_0) \to \pi_n(BPO(\infty)) \to \pi_{n-1}(K(Z_2, 1)) \to \cdots .$$

Now we distinguish the following cases.

i) let $n > 2$, $n \neq 4i$. Bott's theorem gives

$$0 \to 0 \to \pi_n(BPO(\infty)) \to 0$$

so

$$\pi_n(BPO(\infty)) = 0 ,$$

ii) $n = 4i$. We have in this case

$$0 \to Q \to \pi_n(BPO(\infty)) \to 0$$

so

$$\pi_n(BPO(\infty)) = Q.$$

iii) We are left with the cases $n = 2$ and $n = 1$. For these values of $n$, the exact sequence is

$$0 \to \pi_2(BPO(\infty)) \to Z_2 \to 0 \to \pi_1(BPO(\infty)) \to 0$$

so

$$\pi_2(BPO(\infty)) = Z_2$$

and

$$\pi_1(BPO(\infty)) = 0.$$

We have obtained the following results:

$$
\begin{aligned}
&\pi_n(BPO(\infty)) = 0 \qquad &&\text{for } n > 2 \,,\ n \neq 4i \\
&\pi_n(BPO(\infty)) = Q \qquad &&\text{for } n = 4i \\
&\pi_2(BPO(\infty)) = Z_2 \\
&\pi_1(BPO(\infty)) = 0.
\end{aligned}
$$

We must now recall the following

THEOREM (Milnor-Moore [6]). *If an* H-*space has rational homotopy groups, then it is a product of spaces of type* $K(Q, i)$.

Since we know the homotopy groups of the space $BO(\infty)_0$ by Bott's theorem, and as $BO(\infty)_0$ is an H-space, Milnor-Moore's Theorem gives us the following statement

$$BO(\infty)_0 = \prod_{i \geq 1} K(Q, 4i) ,$$

where the symbol $\Pi$ stands for product of spaces. So the fibration (*) can be written in the form

$$(**)\qquad \begin{array}{c} K(Z_2, 1) \\ \downarrow \\ \prod_{i\geq 1} K(Q, 4i) \\ \downarrow \\ BPO(\infty) \end{array}$$

We can now give the following

THEOREM 1.2. BPO $(\infty)$ *is homotopically equivalent to*

$$\prod_{i\geq 1} K(Q, 4i) \times K(Z_2, 2),$$

*that is to say, there is an isomorphism*

$$KPO(-) \cong \bigoplus_{i=1}^{\infty} H^{4i}(-, Q) \oplus H^2(-, Z_2).$$

*Proof.* Take diagram (1.1)

$$\begin{array}{ccc} & K(Z_2, 1) & \to & K(Z_2, 1) \\ & \downarrow & & \downarrow \\ BO(\infty)_0 \cong & \Pi K(Q, 4i) & \to & U \\ & {\scriptstyle\varphi}\downarrow & & \downarrow \\ & BPO(\infty) & \xrightarrow{\psi} & K(Z_2, 2). \end{array}$$

Since $\psi$ induces $\varphi$, it is clear that the homotopic fibre of $\psi$ is just $\prod_{i\geq 1} K(Q, 4i)$. So, if we make $\psi$ into a fibration, $\psi$ is classified by a map

$$\gamma: \quad K(Z_2, 2) \to \Pi K(Q, 4i+1).$$

But $K(Z_2, 2)$ has not rational cohomology, since all its homotopy groups are finite.

So $\gamma$ is homotopically equivalent to the constant map and the fibration

$$BPO(\infty) \to K(Z_2, 2)$$

is trivial. This gives the homotopic equivalence

$$BPO(\infty) \simeq \prod_{i\geq 1} K(Q, 4i) \times K(Z_2, 2).$$

Finally we have.

THEOREM 1.3. $BrO(X) \cong H^2(X, Z_2)$.

*Proof.* By Theorem 1.2, the map

$$BPO(\infty) \simeq \prod_{i\geq 1} K(Q, 4i) \times K(Z_2, 2) \to K(Z_2, 2)$$

is the obvious projection (up to homotopy), so any element of $H^2(X, Z_2)$ is the image of an element of KPO (X).

That is to say the one-to-one map from BrO (X) to $H^2(X, Z_2)$ of Proposition 1.1. is onto.

## § 2. THE ELEMENTS OF ORDER TWO OF Br (X)

The aim of this section is to represent the elements of order 2 of the Brauer group Br (X) of a finite CW-complex X with elements in BrO (X).

For this purpose, consider the diagram

$$\begin{array}{ccc} K(Z_2, 1) & \xrightarrow{\beta} & K(Z_2, 2) \\ \downarrow & & \downarrow \\ BO(\infty)_0 & \rightarrow & BGL(\infty)_0 \\ \downarrow & & \downarrow \\ BPO(\infty) & \rightarrow & BPGL(\infty) \\ \downarrow & & \downarrow \\ K(Z_2, 2) & \xrightarrow{\beta'} & K(Z, 3) \end{array}$$

which is clearly commutative (up to homotopy).

$\beta'$ is the delooping of $\beta$, so it induces the Bockstein map $H^2(X, Z_2) \rightarrow$ $\rightarrow H^3(X, Z)$ corresponding to the exact sequence $Z \rightarrow Z \rightarrow Z_2$. The lowest step of the diagram gives

$$(***) \qquad \begin{array}{ccc} KPO(X) & \xrightarrow{F} & KP(X) \\ \downarrow & & \downarrow \\ H^2(X, Z_2) & \rightarrow & H^3(X, Z) \end{array}$$

where F is the forgetful.

Diagram (***) is commutative. On the other hand, by Section 1, the diagram

$$\begin{array}{ccc} KPO(X) & \xrightarrow{F} & KP(X) \\ \downarrow & & \downarrow \\ BrO(X) & \xrightarrow{F'} & Br(X) \\ \downarrow\wr & & \downarrow\wr \\ H^2(X, Z_2) & \xrightarrow{\xi} & \operatorname{Tor} H^3(X, Z) \end{array}$$

(F′ being the lowered forgetful) is commutative too.

We have thus proved.

THEOREM 2.1. *The diagram*

$$\begin{array}{ccc} BrO(X) & \xrightarrow{F'} & Br(X) \\ {\scriptstyle\alpha}\downarrow\wr & & {\scriptstyle\lambda}\downarrow\wr \\ H^2(X, Z_2) & \xrightarrow{\xi} & \operatorname{Tor} H^3(X, Z) \end{array}$$

*is commutative.*

This tells us that, if $x \in \mathrm{Br}(\mathrm{X})$, and $2x = 0$ (i.e. if $x$ is an element of order 2 of $\mathrm{Br}(\mathrm{X})$), then $\lambda(x) \in \xi(\mathrm{H}^2(\mathrm{X}, \mathrm{Z}))$ and so, if $\xi(y) = \lambda(x)$, with $y \in \mathrm{H}^2(\mathrm{X}, \mathrm{Z}_2)$, then $\mathrm{F}'\alpha^{-1}(y) = x$. Finally we can state

THEOREM 2.2. *The elements of order* 2 *of the* Br (X) *of a finite* CW-*complex* X *may be represented by elements of the orthogonal Brauer* BrO (X).

*Acknowledgment.* The Author is greatly indebted to Prof. Corrado De Concini of the Istituto Matematico of the Università di Pisa, for the conversations we have had about the problem, its material and its approach. Finally I should like to express my thanks to Prof. Claudio Procesi of the Istituto Matematico of the Università di Roma, for carefully reading this paper.

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