Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Yuichi Kitamura, Takaŝi Kusano

Vanishing Oscillations of Solutions of a Class of Differential Systems with Retarded Argument

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **62** (1977), n.3, p. 325–334.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1977_8_62_3_325_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1977.

Equazioni differenziali ordinarie. — *Vanishing Oscillations of Solutions of a Class of Differential Systems with Retarded Argument.* Nota di YUICHI KITAMURA e TAKAŜI KUSANO, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dànno condizioni sufficienti perché tutte le traiettorie oscillatorie del sistema differenziale $x'(t) = p(t) y(t)$, $y'(t) = f(t, x(g(t)))$ tendano a zero quando $t \to \infty$.

1. INTRODUCTION

This paper is concerned with systems of differential equations of the form

$$\text{(A)} \qquad \begin{cases} x'(t) = p(t) y(t), \\ y'(t) = f(t, x(g(t))), \end{cases}$$

where $p(t)$ and $g(t)$ are continuous on $[a, \infty)$ and $f(t, x)$ is continuous on $[a, \infty) \times (-\infty, \infty)$. In addition, it will be assumed throughout that the following conditions hold:

(*a*) $p(t) \geqq 0$, with $p(t)$ not identically zero on any infinite subinterval of $[a, \infty)$;

(*b*) $g(t) \leqq t$, $\lim_{t \to \infty} g(t) = \infty$;

(*c*) $|f(t, x)| \leqq \omega(t, |x|)$ on $[a, \infty) \times (-\infty, \infty)$, where $\omega(t, r)$ is a continuous function on $[a, \infty) \times [0, \infty)$ which is nondecreasing in r and such that $\omega(t, r)/r$ is nonincreasing in r.

If in particular $p(t) > 0$ on $[a, \infty)$, then the system (A) is equivalent to the second order scalar equation

$$\text{(B)} \qquad \left(\frac{1}{p(t)} x'(t)\right)' = f(t, x(g(t))).$$

In what follows our attention will be restricted to solutions $\{x(t), y(t)\}$ of (A) which exist on some ray $[T, \infty)$ and satisfy

$$\sup \{|x(t)| + |y(t)| : t \geqq T'\} > 0 \qquad \text{for any} \quad T' \geqq T.$$

Such a solution is termed *oscillatory* if each of its components is oscillatory in the usual sense, that is, if it has arbitrarily large zeros.

Recently, conditions for all solutions of (A) to be oscillatory have been found by Vareh, Gritsai and Ševelo [4] and by the present authors [1].

(*) Nella seduta del 12 marzo 1977.

The objective of this paper is to study the asymptotic behavior of oscillatory solutions of (A). Sufficient conditions will be given which guarantee that all oscillatory solutions $\{x(t), y(t)\}$ of (A) vanish asymptotically in the sense that $x(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$. When specialized to the scalar equation (B), our results include those of Singh [3] and Kusano and Onose [2].

2. THE CASE $\int_a^\infty p(t)\,dt < \infty$

We begin by examining the differential system (A) in which $p(t)$ satisfies the condition

$$\int_a^\infty p(t)\,dt < \infty.$$

The following notation will be used throughout this section:

$$\pi(t) = \int_t^\infty p(s)\,ds.$$

LEMMA 1. *Assume that*

$$\int^\infty \pi(t)\,\omega(t, 1)\,dt < \infty. \tag{1}$$

If $\{x(t), y(t)\}$ *is a solution of* (A) *defined on* $[T, \infty)$, *then*

$$x(t) = O(1) \quad \text{as} \quad t \to \infty \quad \text{and} \quad p(t)\,y(t) \in L^1[T, \infty).$$

Proof. Let $t_0 \geqq T$ be such that $\tau_0 = \inf\{g(t) : t \geqq t_0\} \geqq T$. By combining

$$|x(t)| \leqq |x(t_0)| + \int_{t_0}^t p(s)\,|y(s)|\,ds, \tag{2}$$

$$|y(t)| \leqq |y(t_0)| + \int_{t_0}^t \omega(s, |x(g(s))|)\,ds, \tag{3}$$

which follow from (A), we have

$$\begin{aligned} |x(t)| &\leqq |x(t_0)| + \pi(t_0)\,|y(t_0)| + \int_{t_0}^t p(s) \int_{t_0}^s \omega(\sigma, |x(g(\sigma))|)\,d\sigma\,ds \\ &\leqq a + \int_{t_0}^t \pi(s)\,\omega(s, |x(g(s))|)\,ds, \qquad t \geqq t_0, \end{aligned} \tag{4}$$

where a is a positive constant. To show that $x(t) = O(1)$ as $t \to \infty$ consider the function

$$u(t) = \max\{1\,,\ \sup_{\tau_0 \leqq s \leqq t} |x(s)|\}\,.$$

We may suppose that $|x(t)| > 1$ for some $t > t_0$. Then, $t_1 > t_0$ can be chosen so that

$$u(t) = \sup_{t_1 \leqq s \leqq t} |x(s)| \qquad \text{for} \quad t \geqq t_1\,. \tag{5}$$

Take $t_2 > t_1$ so large that $g(t) \geqq t_1$ for $t \geqq t_2$ and

$$\int_{t_2}^{\infty} \pi(t)\,\omega(t\,,1)\,dt \leqq \tfrac{1}{2}\,. \tag{6}$$

Using (4), (5) and the condition (c), we obtain

$$\begin{aligned} u(t) &\leqq a + \int_{t_0}^{t} \pi(s)\,\omega(s\,,u(g(s)))\,ds \\ &= b + \int_{t_2}^{t} \pi(s)\,\omega(s\,,u(g(s)))\,ds \\ &\leqq b + u(t)\int_{t_2}^{t} \pi(s)\,\omega(s\,,1)\,ds\,, \qquad t \geqq t_2\,, \end{aligned}$$

where b is a positive constant. In view of (6) this implies $u(t) \leqq 2b$ for $t \geqq t_2$, and so $x(t) = O(1)$ as $t \to \infty$. Multiplying both sides of (3) by $p(t)$, integrating it from t_0 to ∞ and noting that $|x(t)| \leqq c\,, t \geqq t_0$, for some $c \geqq 1$, we see that

$$\int_{t_0}^{\infty} p(t)\,|y(t)|\,dt \leqq \pi(t_0)\,|y(t_0)| + c\int_{t_0}^{\infty} \pi(t)\,\omega(t\,,1)\,dt\,,$$

which shows $p(t)\,y(t) \in L^1[T\,,\infty)$.

The main result of this section is the following

THEOREM 1. *Let $\{x(t)\,, y(t)\}$ be an oscillatory solution of* (A).

(i) *Suppose that* (1) *holds. Then,* $x(t) = o(1)$ *as* $t \to \infty$.

(ii) *Suppose that*

$$\int^{\infty} \omega(t\,,\pi(g(t)))\,dt < \infty\,. \tag{7}$$

Then, $x(t) = o(1)$ *and* $y(t) = o(1)$ *as* $t \to \infty$.

Proof. Let $\{x(t), y(t)\}$ be an oscillatory solution of (A) defined on $[T, \infty)$. Take $t_0 \geqq T$ so that $\tau_0 = \inf\{g(t) : t \geqq t_0\} \geqq T$.

(i) Suppose (1) holds. Then, $p(t)y(t) \in L^1[T, \infty)$ by Lemma 1, and so from the first equation of (A) we have

$$x(t) = x(t_0) + \int_{t_0}^{t} p(s)y(s)\,ds$$
$$= x(t_0) + \int_{t_0}^{\infty} p(s)y(s)\,ds - \int_{t}^{\infty} p(s)y(s)\,ds$$

for $t \geqq t_0$. Since $x(t)$ is oscillatory by hypothesis, we must have

$$x(t_0) + \int_{t_0}^{\infty} p(s)y(s)\,ds = 0, \tag{8}$$

and this yields

$$x(t) = -\int_{t}^{\infty} p(s)y(s)\,ds, \qquad t \geqq t_0. \tag{9}$$

That $x(t) = o(1)$ as $t \to \infty$ is a consequence of (9).

(ii) Suppose (7) holds. Since (7) implies (1), it follows that $x(t) = o(1)$ as $t \to \infty$. Actually, it can be shown that $x(t) = O(\pi(t))$ as $t \to \infty$. Substituting (3) in (9), we find

$$|x(t)| \leqq \pi(t)|y(t_0)| + \int_{t}^{\infty} p(s) \int_{t_0}^{s} \omega(\sigma, |x(g(\sigma))|)\,d\sigma\,ds$$
$$= \pi(t)|y(t_0)| + \pi(t)\int_{t_0}^{t} \omega(s, |x(g(s))|)\,ds$$
$$+ \int_{t}^{\infty} \pi(s)\,\omega(s, |x(g(s))|)\,ds, \qquad t \geqq t_0,$$

which yields

$$|x(t)|/\pi(t) \leqq |y(t_0)| + \int_{t_0}^{t} \omega(s, |x(g(s))|)\,ds$$
$$+ \pi(t)^{-1}\int_{t}^{\infty} \pi(s)\,\omega(s, |x(g(s))|)\,ds, \qquad t \geqq t_0. \tag{10}$$

Suppose to the contrary that $|x(t)|/\pi(t)$ is unbounded and let

$$v(t) = \max\{1\,,\ \sup_{\tau_0 \leqq s \leqq t} [|x(s)|/\pi(s)]\}\,.$$

There exist t_1, t_2, t_3 such that $t_0 < t_1 < t_2 < t_3$,

$$v(t) = \sup_{t_1 \leqq s \leqq t} [|x(s)|/\pi(s)]\,, \qquad t \geqq t_1\,, \tag{11}$$

$$\int_{t_2}^{\infty} \omega(s, \pi(g(s)))\, ds \leqq \frac{1}{4}\,, \tag{12}$$

and

$$|y(t_0)| + v(t_2) \int_{t_0}^{t_2} \omega(s, \pi(g(s)))\, ds \leqq \frac{1}{4} v(t_3)\,. \tag{13}$$

Observing that the right hand side of (10) is an increasing function of t and using (11)-(13), we see that for $t \geqq t_3$

$$\begin{aligned} v(t) &\leqq |y(t_0)| + v(t_2) \int_{t_0}^{t_2} \omega(s, \pi(g(s)))\, ds + v(t) \int_{t_2}^{t} \omega(s, \pi(g(s)))\, ds \\ &\quad + \pi(t)^{-1} \int_{t}^{\infty} \pi(s)\, v(s)\, \omega(s, \pi(g(s)))\, ds \\ &\leqq \tfrac{1}{2} v(t) + \pi(t)^{-1} \int_{t}^{\infty} \pi(s)\, v(s)\, \omega(s, \pi(g(s)))\, ds\,. \end{aligned}$$

Consequently,

$$\pi(t)\, v(t) \leqq 2 \int_{t}^{\infty} \pi(s)\, v(s)\, \omega(s, \pi(g(s)))\, ds\,, \qquad t \geqq t_3\,,$$

from which it follows readily that

$$\sup_{s \geqq t} [\pi(s)\, v(s)] \leqq 2 \sup_{s \geqq t} [\pi(s)\, v(s)] \cdot \int_{t}^{\infty} \omega(s, \pi(g(s)))\, ds\,, \qquad t \geqq t_3\,,$$

or

$$1 \leqq 2 \int_{t}^{\infty} \omega(s, \pi(g(s)))\, ds\,, \qquad t \geqq t_3\,.$$

This contradiction proves that $x(t) = O(\pi(t))$ as $t \to \infty$.

In view of this property of $x(t)$ and (7) we see that $f(t, x(g(t))) \in L^1[t_0, \infty)$. Hence, from the second equation of (A) we get

$$y(t) = y(t_0) + \int_{t_0}^{t} f(s, x(g(s)))\, ds$$

$$= y(t_0) + \int_{t_0}^{\infty} f(s, x(g(s)))\, ds - \int_{t}^{\infty} f(s, x(g(s)))\, ds .$$

Using the fact that $y(t)$ is oscillatory, we have

$$y(t_0) + \int_{t_0}^{\infty} f(s, x(g(s)))\, ds = 0 , \tag{14}$$

and thus we conclude that

$$y(t) = -\int_{t}^{\infty} f(s, x(g(s)))\, ds , \qquad t \geqq t_0 . \tag{15}$$

Therefore, $y(t) = o(1)$ as $t \to \infty$. This completes the proof of Theorem 1.

Example 1. Consider the system

$$\begin{cases} x' = t^{-3/2}(1 + \sin(\ln t))\, y , \\ y' = -(3/2)\, t^{-1} |x|^{1/2} \operatorname{sgn} x + 2\, t^{-3/2}(1 - \cos(\ln t)) . \end{cases} \tag{16}$$

Here we can take

$$\pi(t) = t^{-1/2} \qquad \text{and} \quad \omega(t, r) = (3/2)\, t^{-1} r^{1/2} + 4\, t^{-3/2}.$$

It is easily verified that the condition (7) is satisfied, so that by Theorem 1 —(ii) all oscillatory solutions of (16) vanish asymptotically. Actually, (16) has an oscillatory solution

$$\begin{cases} x(t) = t^{-1}(1 + \sin(\ln t))^2, \\ y(t) = t^{-1/2}(2 \cos(\ln t) - \sin(\ln t) - 1) \end{cases}$$

with this asymptotic property.

3. THE CASE $\int_a^\infty p(t)\,\mathrm{d}t = \infty$

We now turn to the differential system (A) in which $p(t)$ is subject to the condition

$$\int_a^\infty p(t)\,\mathrm{d}t = \infty .$$

The following notation will be used:

$$\mathrm{P}(t) = \int_a^t p(s)\,\mathrm{d}s .$$

LEMMA 2. *Assume that*

$$\int^\infty \omega(t, \mathrm{P}(g(t)))\,\mathrm{d}t < \infty . \tag{17}$$

Then, every solution $\{x(t), y(t)\}$ *of* (A) *has the property*:

$$x(t) = \mathrm{O}(\mathrm{P}(t)) \qquad \textit{and} \quad y(t) = \mathrm{O}(1) \qquad \textit{as} \quad t \to \infty .$$

Proof. Let $\{x(t), y(t)\}$ be defined on $[\mathrm{T}, \infty)$ and let $t_0 \geqq \mathrm{T}$ be such that $\tau_0 = \inf\{g(t) : t \geqq t_0\} \geqq \mathrm{T}$. From (2) and (3) we obtain

$$|x(t)| \leqq |x(t_0)| + \mathrm{P}(t)\,|y(t_0)| + \mathrm{P}(t) \int_{t_0}^t \omega(s, |x(g(s))|)\,\mathrm{d}s$$

for $t \geqq t_0$, which gives

$$|x(t)|/\mathrm{P}(t) \leqq a + \int_{t_0}^t \omega(s, |x(g(s))|)\,\mathrm{d}s , \qquad t \geqq t_0 , \tag{18}$$

where a is a positive constant. Defining

$$u(t) = \max\{1 , \sup_{\tau_0 \leqq s \leqq t} [|x(s)|/\mathrm{P}(s)]\}$$

and applying the same type of argument that was used to prove Lemma 1, we easily conclude from (18) that $u(t)$ is bounded, that is, $x(t) = \mathrm{O}(\mathrm{P}(t))$ as $t \to \infty$. There is a constant $c \geqq 1$ such that $|x(t)| \leqq c\mathrm{P}(t)$ for $t \geqq \mathrm{T}$.

Using this inequality in (3), we have

$$|y(t)| \leqq |y(t_0)| + c\int_{t_0}^{t} \omega(s, P(g(s)))\,ds, \qquad t \geqq t_0,$$

which shows that $y(t) = O(1)$ as $t \to \infty$.

We now state and prove the main result of this section.

THEOREM 2. *Let* $\{x(t), y(t)\}$ *be an oscillatory solution of* (A).

(i) *Suppose that* (17) *holds. Then,* $y(t) = o(1)$ *as* $t \to \infty$.

(ii) *Suppose that*

$$\int^{\infty} P(t)\,\omega(t, 1)\,dt < \infty. \tag{19}$$

Then, $x(t) = o(1)$ *and* $y(t) = o(1)$ *as* $t \to \infty$.

Proof. Let $\{x(t), y(t)\}$ be an oscillatory solution of (A) defined on $[T, \infty)$. Choose $t_0 \geqq T$ so that $\tau_0 = \inf\{g(t) : t \geqq t_0\} \geqq T$.

(i) Suppose (17) holds. By Lemma 2, $x(t) = O(P(t))$ as $t \to \infty$, so that $f(t, x(g(t))) \in L^1[t_0, \infty)$. Integrating the second equation of (A) and noting that (14) holds, we conclude that $y(t)$ admits the expression (15), and hence $y(t) = o(1)$ as $t \to \infty$.

(ii) Suppose (19) holds. Since (19) is stronger than (17), we see that $y(t) = o(1)$ as $t \to \infty$. Substituting (15) in (2), we have

$$\begin{aligned} |x(t)| &\leqq |x(t_0)| + \int_{t_0}^{t} p(s)\int_{s}^{\infty} \omega(\sigma, |x(g(\sigma))|)\,d\sigma\,ds \\ &\leqq |x(t_0)| + \int_{t_0}^{t} P(s)\,\omega(s, |x(g(s))|)\,ds + P(t)\int_{t}^{\infty} \omega(s, |x(g(s))|)\,ds \end{aligned} \tag{20}$$

for $t \geqq t_0$. On the basis of (20) we shall show that $x(t) = O(1)$ as $t \to \infty$. Suppose the contrary. Then, defining

$$v(t) = \max\{1, \sup_{\tau_0 \leqq s \leqq t} |x(s)|\},$$

we can find t_1, t_2, t_3 such that $t_0 < t_1 < t_2 < t_3$, and

$$v(t) = \sup_{t_1 \leqq s \leqq t} |x(s)|, \qquad t \geqq t_1, \tag{21}$$

$$\int_{t_2}^{\infty} P(s)\,\omega(s, 1)\,ds \leqq \frac{1}{4}, \tag{22}$$

$$|x(t_0)| + v(t_2)\int_{t_0}^{t_2} P(s)\,\omega(s, 1)\,ds \leqq \frac{1}{4}\,v(t_3). \tag{23}$$

It is a matter of easy computation to derive from (20)-(23) the following inequality

$$v(t) \leqq 2\,P(t) \int_t^\infty v(s)\,\omega(s, 1)\,ds \qquad \text{for} \quad t \geqq t_3 .$$

From this it follows that

$$\sup_{s \geqq t} [v(s)/P(s)] \leqq 2 \sup_{s \geqq t} [v(s)/P(s)] \cdot \int_t^\infty P(s)\,\omega(s, 1)\,ds , \qquad t \geqq t_3 ,$$

or

$$1 \leqq 2 \int_t^\infty P(s)\,\omega(s, 1)\,ds , \qquad t \geqq t_3 ,$$

which is a contradiction. Therefore, we must have $x(t) = O(1)$ as $t \to \infty$.

Let $c \geqq 1$ be a constant such that $|x(t)| \leqq c$ for $t \geqq T$. Using (15) and this inequality, we have

$$\int_{t_0}^\infty p(s)\,|y(s)|\,ds \leqq \int_{t_0}^\infty p(s) \int_s^\infty \omega(\sigma, |x(g(\sigma))|)\,d\sigma\,ds$$
$$\leqq c \int_{t_0}^\infty p(s)\,\omega(s, 1)\,ds ,$$

and thus $p(t)\,y(t) \in L^1[t_0, \infty)$. If we integrate the first equation of (A) and notice that (8) holds on account of the oscillation of $x(t)$, then we conclude that $x(t)$ admits the expression (9), and therefore $x(t) = o(1)$ as $t \to \infty$. The proof of Theorem 2 is thus complete.

Example 2. Consider the system

$$(24) \quad \begin{cases} x'(t) = 3\,(1 + \sin(\ln t))^2\,y(t) , \\ y'(t) = t^{-\alpha}\,[x^{1/3}(t^\beta) - (1 + \cos(\ln t) + \sin(\ln t) + \sin(\beta \ln t))] , \end{cases}$$

where α and β are positive constants with $\beta \leqq 1$. Here we can take

$$P(t) = t \qquad \text{and} \qquad \omega(t, r) = t^{-\alpha}(r^{1/3} + 4) .$$

If $\alpha > 2$, then (19) is satisfied, so that by Theorem 2—(ii) every oscillatory solution of (24) vanishes asymptotically as $t \to \infty$. If $\alpha = 2$, then (19) is violated, and (24) possesses an oscillatory solution

$$(25) \quad \begin{cases} x(t) = (1 + \sin(\ln t))^3 , \\ y(t) = t^{-1} \cos(\ln t) , \end{cases}$$

not vanishing asymptotically as $t \to \infty$. We observe that the condition (17) is satisfied and the solution (25) justifies the assertion of Theorem 2—(i).

References

[1] Y. Kitamura and T. Kusano – *On the oscillation of a class of nonlinear differential systems with deviating argument.* « J. Math. Anal. Appl. » (to appear).

[2] T. Kusano and H. Onose – *Asymptotic decay of oscillatory solutions of second order differential equations with forcing term.* « Proc. Amer. Math. Soc. » (to appear).

[3] B. Singh (1976) – *Asymptotically vanishing oscillatory trajectories in second order retarded equations,* « SIAM J. Math. Anal. », 7, 37–44.

[4] N. V. Vareh, A. G. Gritsai and V. N. Ševelo (1975) – *On the oscillation of solutions of certain systems of differential equations with retarded argument,* Metody Količestvennogo i Kačestvennogo Issledovaniya Differencial'nyh i Integral'nyh Uravnenii, pp. 20–38, Kiev (Russian).