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Comparison theorems and integrability of nonoscillatory solutions of n-th order functional differential equations

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Equazioni differenziali ordinarie. — *Comparison theorems and integrability of nonoscillatory solutions of n–th order functional differential equations* (*). Nota di LU-SAN CHEN e CHEH-CHIH YEH, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostrano tre teoremi sugli integrali di una equazione funzionale, differenziale, non lineare con argomento ritardato.

1. INTRODUCTION

The results of this note are inspired by two recent papers of Liu [2] and Rankin [4]. The method of these authors can be used to prove three theorems for general n-th order ($n \geq 2$) functional differential non-linear equation with retarded argument

$$(1) \qquad L_n x(t) + [q(t) x(t)]' + p(t) f[x(g(t))] = h(t)$$

where the operators L_i are recursively defined by

$$L_0 x(t) = x(t) \quad , \quad L_i x(t) = \frac{1}{r_i(t)} \frac{d}{dt} L_{i-1} x(t) , \qquad i = 1, 2, \cdots, n,$$

$$r_n(t) = 1 .$$

It is assumed throughout this note that

(i) $r_i(t) \in C[R_+ \equiv [0, \infty), R_+ \setminus \{0\}]$,

$$\int^{\infty} r_i(t)\, dt = \infty , \qquad i = 1, 2, \cdots, n;$$

(ii) $p(t), q(t), g(t), h(t) \in C[R_+, R \equiv (-\infty, \infty)]$,

$$\lim_{t \to \infty} g(t) = \infty \qquad \text{and} \quad q(t) \geq 0;$$

(iii) $f(y) \in C[R, R], yf(y) > 0$ *for* $y \neq 0$.

A nontrivial solution of (1) which exists on $[\tau, \infty)$, for some fixed τ, is called *oscillatory* if it has arbitrary large zeros. Otherwise, it is said to be *nonoscillatory*.

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2. Integrability of nonoscillatory solutions

THEOREM 1. *Let the functions* r_i , p , q , g , h *and* f *satisfy* (i)-(iii) *and, in addition, suppose that*

(C_1) $p(t) \geq M$ *on* $[\tau , \infty)$ *for some constant* $M > 0$,

(C_2) $0 \leq g'(t) \leq 1$ *on* $[\tau , \infty)$,

(C_3) $\int_{\tau}^{\infty} |h(t)| \, dt < \infty$,

(C_4) *There exists a constant* K *such that for* $y \neq 0$

$$\frac{f(y)}{y} \geq K .$$

Then every nonoscillatory solution of (1) *on* $[\tau , \infty)$ *is integrable.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1) on $[\tau , \infty)$. Without loss of generality, we may assume that there is a $T_0 \geq \tau$ such that $x(t) > 0$ for $t \geq T_0$. From (ii) we see that there is a $T_1 \geq T_0$ such that $g(t) \geq T_0$ and $x(g(t)) > 0$ for $t \geq T_1$.

Integrating (1) from T_1 to $t \geq T_1$ and using conditions (C_1)-(C_4), we have

$$\tag{2} L_{n-1} x(t) - L_{n-1} x(T_1) + q(t) x(t) - q(T_1) x(T_1) + \\ + \int_{T_1}^{t} p(s) f[x(g(s))] \, ds \leq \int_{T_1}^{t} |h(s)| \, ds ,$$

or

$$\tag{3} L_{n-1} x(t) - L_{n-1} x(T_1) - q(T_1) x(T_1) + MK \int_{T_1}^{t} x(g(s)) \, ds \leq \int_{T_1}^{t} |h(s)| \, ds .$$

It follows from (C_3) that the left hand side of (3) remains bounded as $t \to \infty$. Suppose $\int_{T_1}^{\infty} x(g(s)) \, ds = \infty$. Then $\lim_{t\to\infty} L_{n-1} x(t) = -\infty$. This and (i) imply that $\lim_{t\to\infty} x(t) = -\infty$, a contradiction. Therefore, we have

$$\tag{4} \int_{T_1}^{\infty} x(g(s)) \, ds < \infty .$$

By (C_2) and change of variable, we get

$$\infty > \int_{T_1}^{\infty} x(g(s))\, ds \geq \int_{T_1}^{\infty} g'(s)\, x(g(s))\, ds = \int_{g(T_1)}^{\infty} x(t)\, dt. \tag{5}$$

Hence the proof is complete.

Remark 1. Taking $r_{n-1}(t) = r(t)^{-1}$, $r_i(t) \equiv 1$, $i = 1, 2, \cdots, n-2$ and $f(x) \equiv x$, then Liu's result [2] is a special case of our theorem.

Remark 2. For the case $n = 2$, $r_1(t) \equiv 1$, $f(x) \equiv x$ and $q(t) \equiv 0$, the equation (1) is studied by Dahiya and Singh [1]. In the assumptions of our theorem, we do not require the condition $p(t)\, p''(t) \leq 2\, (p'(t))^2$ and the boundedness of the delay term $\tau(t) = t - g(t)$ which are assumed in [1].

3. COMPARISON THEOREMS

In this section, we compare the following two equations:

$$L_n x(t) + p(t)\, x(g(t)) = h(t), \tag{5}$$

$$L_n x(t) + p(t)\, x(g(t)) = 0. \tag{7}$$

A solution $x(t)$ of equation (6) (resp. 7) is termed *oscillatory* if it has an increasing sequence of zeros $\{t_n\}$ on $[\tau, \infty)$ such that $\lim_{n\to\infty} t_n = \infty$.

Equation (6) (resp. 7) is called *oscillatory* if all its solutions are oscillatory. Equation (6) (resp. 7) is called *nonoscillatory* if every solution is nonoscillatory.

THEOREM 2. *Assume that the equation* (7) *is oscillatory, then every pair of nonoscillatory solutions of equation* (6) *eventually has the same sign.*

Proof. Suppose $x(t)$ and $y(t)$ are nonoscillatory solutions of (6), then $W(t) = x(t) - y(t)$ is a solution of (7).

By hypothesis there exists an infinite sequence $\{t_n\}$ of zeros of $W(t)$. Hence $x(t_n) = y(t_n)$ for all n, implying that $x(t)$ and $y(t)$ have the same sign for large t.

Recently, Lovelady [3] proved the following theorem.

THEOREM 3. *If*

$$\int^{\infty} \omega_{n-1}(t)\, p(t)\, dt = \infty \tag{8}$$

where $\omega_i(t) \in C[R_+, R_+]$, $i = 1, 2, \cdots, n-1$ *are defined by*

$$\omega_1(t) = \int_0^t r_1(s)\, ds \quad , \quad \omega_k(t) = \int_0^t r_k(s)\, \omega_{k-1}(s)\, ds, \qquad k = 2, 3, \cdots, n-1.$$

then every bounded solution of the equation

$$\mathrm{L}_n x(t) + (-1)^n p(t) x(g(t)) = 0 \,, \qquad (p(t) > 0) \tag{9}$$

is oscillatory.

From Theorems 2 and 3, we easily see that the following holds.

COROLLARY. *Let condition* (8) *hold. Then every pair of nonoscillatory bounded solutions of the equation*

$$\mathrm{L}_n x(t) + (-1)^n p(t) x(g(t)) = h(t) \,, \qquad (p(t) > 0) \tag{10}$$

eventually has the same sign.

Remark 3. Taking $n = 2$, $r_1(t) = r(t)^{-1}$ and $g(t) \equiv t$, then Rankin's results [4, Theorem 9] is a special case of our result [Theorem 2].

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