Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On the mean convergence of Dini series

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **62** (1977), n.3, p. 305–315.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1977_8_62_3_305_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1977.

Funzioni speciali. — *On the mean convergence of Dini series.* Nota di S. R. AGRAWAL e C. M. PATEL, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questo lavoro si prova che un sistema ortogonale di funzioni di Bessel è una base nello spazio di Banach $L_\beta^p(0, 1)$, $p > 1$, $-1 < \beta < p - 1$. Se ne deduce che la serie di Dini di ogni funzione $f \in L_\beta^p$ converge a f nella norma di L_β^p. Inoltre si dimostra, tramite un controesempio, che se la condizione $1 < \beta < p - 1$ non è soddisfatta esiste una funzione di questa classe la cui serie di Dini diverge.

1. INTRODUCTION

Let $L_\beta^p(0, 1)$, $p \geq 1$, β any real number, be the class of all measurable functions f defined on $[0, 1]$, for which

$$\|f\|_{p,\beta} = \left\{ \int_0^1 |f(x)|^p x^\beta \, dx \right\}^{1/p} < \infty . \tag{1.1}$$

It is known, e.g., Alexits [1], that L_β^p is a Banach space under the norm defined by (1.1). It is easy to verify that $L_\alpha^p \subseteq L_\beta^p$, for $\alpha \leq \beta$; $L_0^p = L^p$ and $L_\beta^p(a, b) = L^p(a, b)$, for $0 < a < b$.

For $p > 1$, $1/p + 1/p' = 1$, $\beta' = -\beta/(p-1)$, $L_{\beta'}^{p'}(0, 1)$ denotes the conjugate space for $L_\beta^p(0, 1)$. If $f \in L_\beta^p$ and $g \in L_{\beta'}^{p'}$, then $fg \in L^1$ and

$$\left| \int_0^1 f(x) \, g(x) \, dx \right| \leq \|f\|_{p,\beta} \, \|g\|_{p',\beta'} . \tag{1.2}$$

The following properties of the class L_β^p have been established in Gol'dman [3] ($K_1, K_2, K_3, \cdots$ etc. denote constants depending upon p, β and ν only):

1.1. LEMMA. *Let* $f \in L_\beta^p(0, 1)$, $p > 1$, $\beta < p - 1$; *and let* $F(x) = \int_0^x |f(t)| \, dt$. *Then* $\frac{F(x)}{x} \in L_\beta^p(0, 1)$ *and*

$$\left\| \frac{F(x)}{x} \right\|_{p,\beta} \leq K_1 \|f\|_{p,\beta} . \tag{1.3}$$

(*) Nella seduta del 12 marzo 1977.

1.2. LEMMA. *Let* $f \in L_\beta^p(0, 1)$, $p > 1$, $-1 < \beta < p - 1$; *and let* $G(x) = \int_0^1 \frac{|f(t)|}{x+t} dt$. *Then* $G \in L_\beta^p(0, 1)$ *and*

$$\|G\|_{p,\beta} \leq K_2 \|f\|_{p,\beta}. \tag{1.4}$$

1.3. LEMMA. *If* $f \in L^p(0, 1)$, $p > 0$, *and* $g(x) = \int_0^1 \frac{|f(t)|}{2 - x - t} dt$, *then* $g \in L^p(0, 1)$ *and* $\|g\|_p \leq K_3 \|f\|_p$.

2. DINI SERIES

Let

$$\begin{cases} \varphi_m(t) = e_m^{-1} \sqrt{t}\, J_\nu(t\lambda_m), & t > 0, \\ \varphi_m(0) = \lim_{t \to 0} \varphi_m(t), \end{cases} \tag{2.1}$$

where $J_\nu(t)$ is the Bessel function of the first kind for $\nu \geq -1/2$; $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ the successive positive zeros of $lt\, J_\nu'(t) + h J_\nu(t)$ and

$$2\lambda_m^2 e_m^2 = (\lambda_m^2 - \nu^2) J_\nu^2(\lambda_m) + \lambda_m^2 J_\nu'^2(\lambda_m). \tag{2.2}$$

Since $J_\nu(t) \sim \frac{t^\nu}{\Gamma(\nu + 1)}$ near zero, $\varphi_m \in L_\beta^p(0, 1)$, if $\nu + \frac{1}{2} > 0$ for $\beta > -1$ and $(\nu + 1/2)p + \beta + 1 > 0$ for $\beta \leq -1$.

Gol'dman [3] has shown that $\{\sqrt{2x}\, J_{\nu+1}^{-1}(j_m) J_\nu(xj_m) : m = 1, 2, \cdots\}$ where $j_1 < j_2 < j_3 < \cdots$ are the successive positive zeros of $J_\nu(t)$, forms an orthonormal basis in the space $L_\beta^p(0, 1)$ for $p > 1$ and $-1 < \beta < p - 1$. He has also established the divergence of Fourier-Bessel series of functions in L_β^p corresponding to this system when $\beta \geq p - 1$ or $\beta \leq -1$.

In the present paper, we establish similar results regarding the orthonormal system $\{\varphi_m(t)\}$ corresponding to the class L_β^p. We also prove the convergence in the L_β^p-norm of the sequence of partial sums of Dini series, corresponding to $f \in L_\beta^p$, given by

$$f(x) \sim \sum_{m=1}^{\infty} b_m \varphi_m(x), \qquad (h/l + \nu > 0 \quad \text{or} \quad l \neq 0), \tag{2.3}$$

where $b_m = \int_0^1 f(t) \varphi_m(t) dt$.

These partial sums are given by

$$S_n(x, f) = \int_0^1 f(t)\, U_n(t, x)\, dt, \tag{2.4}$$

where

$$U_n(t, x) = \sum_{m=1}^{n} \frac{\sqrt{xt}\; J_\nu(t\lambda_m)\, J_\nu(x\lambda_m)}{e_m^2}. \tag{2.5}$$

If $\lambda_n < D_n < \lambda_{n+1}$, then by Watson ([8]; p. 602),

$$U_n(t, x) = \frac{1}{2i} \int_{D_n - i\infty}^{D_n + i\infty} \frac{\sqrt{tx}\; w\Phi(w, x)\, J_\nu(tw)}{lw\, J'_\nu(w) + h J_\nu(w)}\, dw, \tag{2.6}$$

when $0 < t < x < 1$, $Y_\nu(t)$ is the Bessel function of 2nd kind and

$$\left\{\begin{aligned} \Phi(w, x) &= (h + lw)\, \Phi_1(w, x) - w\Phi_2(w, x), \\ \Phi_1(w, x) &= J_\nu(w)\, Y_\nu(xw) - J_\nu(xw)\, Y_\nu(w), \qquad \text{and} \\ \Phi_2(w, x) &= J_{\nu+1}(w)\, Y_\nu(xw) - J_\nu(xw)\, Y_{\nu+1}(w). \end{aligned}\right. \tag{2.7}$$

In case, $0 < x < t < 1$, t and x are to be interchanged in (2.6).
Moore [5] has shown that

$$\lambda_n = n\pi + q + K_4(\lambda_n)/n, \tag{2.8}$$

where

$$q = \begin{cases} k\pi + (2\nu + 1)\pi/4, & \text{if } l \neq 0, \\ k\pi + (2\nu - 1)\pi/4, & \text{if } l = 0, \end{cases}$$

k is an integer and $K_4(\lambda_n)$ remains bounded for large n.
We may, therefore, choose for large n,

$$D_n = n\pi + (2\nu + 3)\pi/4. \tag{2.9}$$

3. Evaluation of the spectral function $U_n(t, x)$

For any $x, t \in (0, 1)$, $x \neq t$, let a positive integer n be chosen so that

$$D_n > K_5/(x - t). \tag{3.1}$$

3.1. LEMMA. *Let* $0 < x < 2/D_n$, $0 < t < 4/D_n$. *Then*

$$|U_n(t, x)| \leq K_6 D_n. \tag{3.2}$$

Proof. Let $t < x$. By Watson ([8]; §§ 3.61 and 18.51), for w lying on $D_n - i\infty$ to $D_n + i\infty$,

$$|\Phi(w, x)| \leq \frac{K_7 e^{(1-x)|v|}}{\sqrt{x}}. \tag{3.3}$$

Again, from the recurrence relations and asymptotic formulae, Watson ([8]; §§ 3.2 and 7.21),

$$\left| \frac{w}{h J_\nu(w) + lw J'_\nu(w)} \right| \leq K_8 \sqrt{|w|}\, e^{-|v|}. \tag{3.4}$$

Therefore, using (2.6) and (3.1), we get

$$|U_n(t, x)| \leq K_5 D_n.$$

If $x < t$, (3.2) is obtained by interchanging t and x.

3.2. LEMMA. *For* $2/D_n < x < 1$, $1/D_n < t < 1$,

$$U_n(t, x) = 1 + O(D_n^{-1})\, \bar{U}_n(t, x), \tag{3.5}$$

where

$$\bar{U}_n(t, x) = \frac{1}{2}\left\{ \frac{\sin D_n(x - t)}{\sin \pi/2\,(x - t)} + \frac{\sin D_n(2 - x - t)}{\sin \pi/2\,(x + t)} \right\} + \tag{3.6}$$

$$+ O\left(\frac{1}{D_n\, xt\,(2 - x - t)} \right) + O(1).$$

Proof. By Watson ([8]; § 7.21),

$$\frac{1}{h J_\nu(w) + lw J'_\nu(w)} = \frac{1}{-l\sqrt{2w/\pi}\, \cos(w - \nu\pi/2 - 3\pi/4) + O(|w|^{-1})}.$$

Therefore, if $t < x$, by (2.6), (3.5) is true for

$$\bar{U}_n(t, x) = \frac{1}{2il} \int_{D_n - i\infty}^{D_n + i\infty} \sqrt{\pi w t x/2}\; \frac{\Phi(w, x)\, J_\nu(tw)\, dw}{\cos(w - \nu\pi/2 + \pi/4)}, \qquad l \neq 0. \tag{3.7}$$

Again, using the following asymptotic expansions in (2.7),

$$J_\nu(z) = \sqrt{2/\pi z}\, \{P_z \cos(z - a) - Q_z \sin(z - a)\}$$

and

$$Y_\nu(z) = \sqrt{2/\pi z}\, \{P_z \sin(z - a) + Q_z \cos(z - a)\},$$

where

$$|z| > 1 , a = (2\nu + 1)\pi/4$$

and

$$P_z = 1 + O(|z|^{-2}) \quad , \quad Q_z = (K_9/z)\{1 + O(|z|^{-2})\},$$

we obtain from (3.7),

$$\bar{U}_n(t, x) = \frac{K_{10}}{\pi i} \int_{D_n - i\infty}^{D_n + i\infty} \left[- \frac{\cos(1-x)w\cos(tw-a)}{\sin(w-a)} + \right.$$

$$+ \left\{ \frac{\sin(1-x)w\cos(tw-a)}{xw\sin(w-a)} - \frac{\cos(1-x)w\sin(tw-a)}{tw\sin(w-a)} \right\} -$$

$$\left. - \frac{h + l\nu}{l} \cdot \frac{\sin(1-x)w\cos(tw-a)}{w\sin(w-a)} + \frac{\mu(t, x, w)}{t^2 w^2} \right] dw ,$$

(3.8) $$= I_1 + I_2 + I_3 + I_4 , \quad \text{say},$$

where $|\mu(t, x, w)| \leq e^{-(x-t)|v|}$, for w lying on $D_n + i\infty$ to $D_n - i\infty$.

Now, if $1/D_n < t < x/2 , x - t > x/2$, so that

(3.9) $$|I_4| < \frac{K_{11}}{D_n xt} .$$

If $x/2 < t < x$, then (3.9) is true from (3.1). Hence (3.9) is true for $1/D_n < t < x < 1$.

Further,

(3.10) $$|I_2| \leq \frac{K_{10}(x+t)}{\pi xt} \int_0^\infty \frac{e^{(|1-x-t|-1)v}}{\sqrt{D_n^2 + v^2}} dv + \frac{K_{10}(x-t)}{\pi xt} \int_0^\infty \frac{e^{(|1-x+t|-1)v}}{\sqrt{D_n^2 + v^2}} dv$$

$$= O\left(\frac{1}{xt D_n (2 - x - t)} \right) .$$

In a similar way,

(3.11) $$I_3 = O\left(\frac{1}{xt D_n (x+t)(2-x-t)} \right) + O(1) .$$

Finally, using the integral,

$$\int_{-\infty}^{\infty} \frac{\cos h\lambda v}{\cos h v} dv = \frac{\pi}{\sin \pi/2(1-\lambda)} , \qquad -1 < \lambda < 1 ,$$

we obtain, in view of (2.9),

$$I_1 = \frac{\sin D_n (x - t)}{2 \sin \pi/2 (x - t)} + \frac{\sin D_n (2 - x - t)}{2 \sin \pi/2 (x + t)} . \tag{3.12}$$

The lemma, now, follows for $t < x$, from (3.8) to (3.12). For $x < t$, t and x are merely to be interchanged.

3.3. LEMMA. *For* $2/D_n < x < 1$ *and* $0 < t < 1/D_n$,

$$| U_n (t , x) | \le K_{12} t^{\nu+1/2} x^{-1} D_n^{\nu+1/2} \le K_{12}/x , \tag{3.13}$$

and for $0 < x < 2/D_n$, $4/D_n < t < 1$,

$$| U_n (t , x) | \le K_{13} x^{\nu+1/2} t^{-1} D_n^{\nu+1/2} . \tag{3.14}$$

Proof. (3.13) follows from Watson ([8]; § 3.31), (2.6), (3.3) and (3.4). (3.14) follows in a similar way.

4. MAIN THEOREMS

4.1. THEOREM. *The system* $\{\varphi_m(t)\}$ *forms a basis in the Banach space* $L_\beta^p(0 , 1)$, *where* $p > 1$, $-1 < \beta < p - 1$ *and* $\nu \ge -1/2$.

Proof. By Pollard ([6]; p. 361), it is enough to show that

$$\| S_n (x , f) \|_{p,\beta} \le K_{14} \| f \|_{p,\beta} , \qquad \text{for all} \quad f \in L_\beta^p . \tag{4.1}$$

Let $0 < x < 2/D_n$. Then by (2.4), (3.2), (1.2) and (3.14),

$$| S_n (x , f) | \le K_6 D_n \int_0^{4/D_n} | f(t) | \, dt + \tag{4.2}$$

$$+ K_{13} x^{\nu+1/2} D_n^{\nu+1/2} \int_{4/D_n}^{1} \frac{| f(t) |}{t} \, dt \le K_{15} D_n^{(1+\beta)/p} \| f \|_{p,\beta} .$$

Therefore, we have

$$\int_0^{2/D_n} | S_n (x , f) |^p x^\beta \, dx \le K_{16}^p \| f \|_{p,\beta}^p . \tag{4.3}$$

Now, let $2/D_n < x < 1$. Then

$$S_n (x , f) = \left(\int_0^{1/D_n} + \int_{1/D_n}^{1} \right) f(t) \, U_n (t , x) \, dt = \tag{4.4}$$

$$= \sigma_1 + \sigma_2 , \qquad \text{say.}$$

By (3.13) and Lemma 1.1, we obtain, as in (4.3),

$$\int_{2/D_n}^{1} |\sigma_1(x)|^p x^\beta \, dx \leq K_{17}^p \|f\|_{p,\beta}^p . \tag{4.5}$$

Further,

$$\sigma_2(x) = \frac{1}{2} \int_{1/D_n}^{1} \frac{\sin D_n (x - t)}{\sin \pi/2 \, (x - t)} f(t) \, dt + \tau(x) , \tag{4.6}$$

where

$$|\tau(x)| \leq \frac{1}{2} \left| \int_{1/D_n}^{1} \frac{\sin D_n (2 - x - t)}{\sin \pi/2 \, (x + t)} f(t) \, dt \right| + \tag{4.7}$$

$$+ K_{18} \int_{1/D_n}^{1} \frac{|f(t)|}{xt \, D_n (2 - x - t)} \, dt + K_{19} \int_{1/D_n}^{1} |f(t)| \, dt =$$

$$= \tau_1(x) + \tau_2(x) + \tau_3(x) , \qquad \text{say} .$$

Now, using Lemma 1.2, we get

$$\int_{2/D_n}^{1} |\tau_1(x)|^p x^\beta \, dx \leq K_{20}^p \|f\|_{p,\beta}^p . \tag{4.8}$$

Also,

$$\tau_2(x) \leq \frac{2 K_{18}}{x D_n} \int_{1/D_n}^{1/2} \frac{|f(t)|}{t} \, dt + K_{18} \int_{1/2}^{1} \frac{f(t)}{2 - x - t} \, dt = \tag{4.9}$$

$$= \tau_2'(x) + \tau_2''(x) , \qquad \text{say} .$$

As in (4.3),

$$\int_{2/D_n}^{1} |\tau_2'(x)|^p x^\beta \, dx \leq K_{21}^p \|f\|_{p,\beta}^p ; \tag{4.10}$$

and by Lemma 1.3,

$$\int_{2/D_n}^{1} |\tau_2''(x)|^p x^\beta \, dx = K_{18}^p \left\{ \int_{2/D_n}^{1/2} + \int_{1/2}^{1} \right\} \left| \int_{1/2}^{1} \frac{f(t)}{2 - x - t} \, dt \right|^p x^\beta \, dx \tag{4.11}$$

$$\leq K_{22}^p \|f\|_1^p + K_{23}^p \int_{1/2}^{1} |g(x)|^p \, dx$$

$$\leq K_{24}^p \|f\|_{p,\beta}^p .$$

Similarly,

$$\int_{2/D_n}^{1} |\tau_3(x)|^p x^\beta \, dx \leq K_{25}^p \|f\|_{p,\beta}^p . \tag{4.12}$$

We also have, for $2/D_n < x < 1$, $0 < t < 1/D_n$, $x - t > x/2$, so that

$$|I(x)| = \left| \frac{1}{2} \int_0^{1/D_n} \frac{\sin D_n(x-t)}{\sin \pi/2\,(x-t)} f(t)\, dt \right| \leq K_{26} \frac{F(x)}{x} .$$

Therefore,

$$\int_{2/D_n}^{1} |I(x)|^p x^\beta \, dx \leq K_{27}^p \|f\|_{p,\beta}^p . \tag{4.13}$$

Thus, from (4.4) to (4.13), it follows that for $2/D_n < x < 1$,

$$S_n(x, f) = \frac{1}{2} \int_0^1 \frac{\sin D_n(x-t)}{\sin \pi/2\,(x-t)} f(t)\, dt + \bar{S}_n(x) , \tag{4.14}$$

where

$$\int_{2/D_n}^{1} |\bar{S}_n(x)|^p x^\beta \, dx \leq K_{28}^p \|f\|_{p,\beta}^p . \tag{4.15}$$

Using (2.9) if we set $\gamma = (\nu + 1/2)\pi/2$, we obtain,

$$\begin{aligned} \frac{1}{2} \int_0^1 \frac{\sin D_n(x-t)}{\sin \pi/2\,(x-t)} f(t)\, dt &= \cos \gamma x \int_0^1 \Delta_n(x, t) f(t) \cos \gamma t \, dt + \\ &+ \sin \gamma x \int_0^1 \Delta_n(x, t) f(t) \sin \gamma x \, dt + \\ &+ \int_0^1 \frac{\sin \gamma(x-t)}{2 \sin \pi/2\,(x-t)} f(t) \cos \{(n + \tfrac{1}{2})\pi(x-t)\}\, dt , \end{aligned} \tag{4.16}$$

where $\Delta_n(x, t)$ is the Dirichlet kernel for Fourier series.

From Hardy-Littlewood [4] and Babenko [2], it is true that the functions of the class $L_\beta^p(0, 1)$ can be expanded by trigonometric systems. Hence, the partial sums of the Fourier series with respect to these systems satisfy the inequality (4.1).

Hence, from (4.3) and (4.14) to (4.16), (4.1) is proved.
The theorem is, thus, completely proved.

4.11. *Remark.* It may be noted that the system discussed by Gol'dman [3] is a special case of our system (2.1), when $l = 0$. Also, when $\nu = \pm \frac{1}{2}$, (2.1) reduces to a trigonometric system.

4.2. THEOREM. *For* $\nu \geq -1/2$ *and* $f \in L_\beta^p$, $p > 1$, $-1 < \beta < p - 1$,

$$\lim_{n\to\infty} \int_0^1 |f(x) - S_n(x)|^p x^\beta \, dx = 0 . \tag{4.17}$$

Proof. From Young ([9]; § 9), it is true that for any function f, if

$$\int_0^1 f(t)\, \varphi_m(t)\, dt = 0 , \qquad m = 1, 2, 3, \cdots,$$

then f is a null function. From this and Theorem 4.1, it follows that the system $\{\varphi_m(t)\}$ forms a complete orthonormal basis in $L_\beta^p(0, 1)$, hence, it is closed in L_β^p.

(4.17) follows from the above conclusion.

5. DIVERGENCE OF DINI SERIES

We shall, now, exhibit an example to prove the condition $-1 < \beta < p - 1$ to be essential for the convergence of the Dini series (2.3) corresponding to $f \in L_\beta^p$.

Gol'dman [3] has proved the following lemma:

5.1. LEMMA. *Let* $\nu > -1/2$. *Then the following relations hold*:

$$\int_0^1 | J_\nu(x\lambda_m)\, x^{1/2} |^p x^\beta \, dx \sim \lambda_m^{-p/2} \log \lambda_m , \qquad \beta = -1 , p \geq 1 , \tag{5.1}$$

and

$$\int_0^1 | J_\nu(x\lambda_m)\, x^{1/2} |^p x^\beta \, dx \sim \lambda_m^{-p/2} , \qquad \beta > -1 , p \geq 1 , \tag{5.2}$$

where $\sim$ *denotes the presence of bilateral relations with constants depending upon* ν, p *and* β *only*, m *being sufficiently large.*

5.2. LEMMA. *For $\nu > -1/2$, m sufficiently large, the following estimates are true*:

$$\| \varphi_m \|_{p,\beta} \sim 1 , \qquad \beta > -1 , p \geq 1 , \tag{5.3}$$

and

$$\| \varphi_m \|_{p,\beta} \sim (\log \lambda_m)^{1/p} , \qquad \beta = -1 , p \geq 1 . \tag{5.4}$$

Proof. By recurrence relations and asymptotic expansions,

$$\begin{aligned} e_m^2 &= \tfrac{1}{2} \{ J_\nu^2 (\lambda_m) + J_{\nu+1}^2 (\lambda_m) \} - (\nu/\lambda_m) J_\nu (\lambda_m) J_{\nu+1} (\lambda_m) = \\ &= 1/(\pi\lambda_m) \{ 1 + O (\lambda_m^{-1}) \} . \end{aligned} \tag{5.5}$$

Hence, (5.3) follows from (5.2) and (5.5). Similarly, (5.4) follows from (5.1) and (5.5).

5.3. THEOREM. *There exists a function $f_0 \in L_{p-1}^p$, whose Dini series diverges in the L_{p-1}^p–norm.*

Proof. For each m, $m = 1, 2, 3, \cdots$ etc., $\varphi_m \in L_{-1}^{p'}$, the space conjugate to L_{p-1}^p, and hence defines a linear functional on the space L_{p-1}^p. Thus, by (5.4), $\{\varphi_m\}$ is a sequence of linear functionals whose norms form an unbounded set. By the Banach-Steinhaus theorem, Taylor ([7]; § 4.4), there exists a function $f_0 \in L_{p-1}^p$, such that the Dini coefficients corresponding to f_0 do not form a bounded sequence.

Hence, the general term of the Dini series does not tend to zero in the L_{p-1}^p-norm. This proves the theorem.

5.31. *Remark.* For $\beta \geq p - 1$, to have $\varphi_m \in L_\beta^p (0, 1)$, we must have $(\nu + 1/2) p + (p - 1) - \beta > 0$.

5.4. THEOREM. *For $\beta > p - 1$, $p > 1$, there is a function in $L_\beta^p (0, 1)$, whose Dini series diverges.*

Proof. For $\beta > p - 1$, $L_{p-1}^p \subseteq L_\beta^p$. Hence, the theorem follows from Theorem 5.3.

5.5. THEOREM. *For $\beta \leq -1$, $p > 1$, there is a function in $L_\beta^p (0, 1)$, whose Dini series diverges.*

Proof. For $\beta \leq -1$, either $\{\varphi_m\}$ does not form a basis for L_β^p or it forms a basis for $L_{\beta'}^{p'}$. If it forms a basis for $L_{\beta'}^{p'}$, then for $\beta \leq -1$, $\beta' \geq p' - 1$. Therefore, the theorem follows from Theorems 5.3 and 5.4.

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