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On the stability of spherically symmetric flows

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Meccanica dei fluidi. — *On the stability of spherically symmetric flows* (*). Nota di ANDREA PROSPERETTI, presentata (**) dal Corrisp. P. CALDIROLA.

RIASSUNTO. — Si considera il moto prodotto in un liquido lievemente viscoso da una cavità di forma approssimativamente sferica e di raggio variabile. La parete della cavità è descritta in termini di uno sviluppo in armoniche sferiche i cui coefficienti sono funzioni del tempo. Nella ipotesi di piccola distorsione della forma sferica si deriva una equazione di moto linearizzata per tali quantità e si discutono varie caratteristiche del risultato ottenuto.

The question of the stability of the interface between two superposed fluids of different densities is a classical problem in fluid mechanics, to which the names of Rayleigh and Taylor are usually associated. Indeed, although a treatment of waves at the interface between two fluids of different densities can be found in a famous paper by G. G. Stokes, published in 1847 [1], it was Lord Rayleigh [2, 3] who recognized the implications of Stoke's result for the stability problem, discussing it with much greater generality, and G. I. Taylor [4] who clarified the effect of a superimposed acceleration. The results are well-known, and we shall give here only their limiting form as the density of one of the fluids becomes vanishingly small so that one deals, in fact, with only one liquid. If the displacement of the interface from the (plane) equilibrium configuration is in the form of a sinusoidal wave of wavenumber k, in the linearized approximation the time dependence of its amplitude, $a(t)$, is governed by the following equation

$$\frac{d^2 a}{dt^2} - \left(\hat{g}\, k - \frac{\sigma}{\rho}\, k^3\right) a = 0. \tag{1}$$

Here $\hat{g}$ is the "effective" acceleration defined by $\hat{g} = g_1(t) - g$, where $g_1(t)$ is an imposed, possibly time-varying, acceleration and g is the acceleration of gravity; both g and $g_1(t)$ should be considered positive when they are directed into the liquid. The surface tension has been denoted by σ, and the liquid density by ρ.

It is clear that, for positive constant $\hat{g}$ (for example, liquid lying above the free surface, with $g_1 = 0$), Eq. (1) predicts an unbounded growth of perturbations the wave number of which is smaller than a "cutoff" wave number k_c given by $k_c^2 = \rho\hat{g}/\sigma$. In this case therefore the configuration is unstable whenever the wavelenght of the allowable perturbations is not suitably restric-

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ted, for instance by the presence of rigid boundaries. On the other hand, when $\hat{g}$ is negative, one has unconditional stability. Qualitatively similar results hold also for time-varying $\hat{g}$. The situation may be summarized by the statement that the plane interface is stable or unstable according as the " effective " acceleration is directed away from or into the liquid respectively.

Equation (1) is valid only insofar as viscous effects can be neglected. The viscous case has been treated by several authors [5-7] in the framework of a normal-mode analysis, the limitations of which have however recently been pointed out by us [8]. In the same study it was nevertheless shown that, in the limit of small viscosity, the complete equation (which has an integro-differential structure) can be approximated by adding a dissipative term to Eq. (1). This correction can also be obtained either by taking the small-viscosity limit of the normal-mode result, or by making use of Stokes' dissipation function which, to first order in viscous effects, can be computed from the inviscid potential flow solution (Ref. [9], Section 348; Ref. [10], Section 25). The approximate equation which is found in this way is

$$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} + 4\,\nu k^2 \frac{\mathrm{d}a}{\mathrm{d}t} - \left(\hat{g}\,k - \frac{\sigma}{\rho}\,k^3\right) a = 0\,, \tag{2}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the liquid. In view of the following developments we would like to note explicitly an important feature of this equation, namely that viscosity merely introduces a term proportional to $\mathrm{d}a/\mathrm{d}t$, without any effect on that proportional to a. Interpreting Eq. (2) as the equation of a fictitious oscillator, we may say that viscosity introduces damping, but has no effect on the restoring force. This fact has the important consequence that the stability features found in the inviscid case are not modified, but only the rate of growth or decay of the amplitude of the disturbances is affected. One of the purposes of this note is to show that, in the case of a slightly distorted spherical interface of variable radius, this characteristic is no longer conserved: viscosity affects also the " restoring force ", and therefore unavoidably modifies the stability properties of the configuration. It appears likely that such a feature is not characteristic only of the spherical situation, but is to be encountered with any non-plane free surface.

We consider a nearly spherical cavity (bubble) of time-varying mean radius $\mathrm{R}\,(t)$ in an unbounded viscous liquid. In terms of spherical coordinates centered at the centroid of the cavity we may describe the shape of the free surface by means of an expansion in spherical harmunics as follows

$$r\,(\theta\,,\varphi) = \mathrm{R}\,(t) + \sum_{l,m} a_{lm}\,(t)\,\mathrm{Y}_l^m\,(\theta\,,\varphi)\,.$$

In the linenarized theory it is found that the equations for the different a's are uncoupled, and furthermore that they are inepedndent of the degree m of the spherical harmonic [11-13]. Accordingly, we shall omit the second index from the amplitudes, and we shall restrict our attention to only one of them, a_n say. Further, as is appropriate for a bubble filled with an ordinary gas or vapor far from the critical point, we shall neglect in this work the dynamical effects of the fluid contained in the cavity; they are considered, however, in Ref. [13]. Let us remark here that we adopt the point of view that the mean radius $R(t)$ is a prescribed function of time. For any particular problem this quantity may be obtained from the well-known Rayleigh-Plesset equation [11, 14], which can be solved independently of the quantities a_l.

We have shown elsewhere [13] that in the hypotheses stated above the amplitude $a_n(t)$ of the n-th order spherical harmonic is given by the following equation

$$\frac{d^2 a_n}{dt^2} + \left[3\frac{1}{R}\frac{dR}{dt} - 2(n-1)(n+1)(n+2)\frac{\nu}{R^2}\right]\frac{da_n}{dt} \tag{3}$$

$$+ (n-1)\left[-\frac{1}{R}\frac{d^2 R}{dt^2} + 2(n+1)(n+2)\frac{\nu}{R^3}\frac{dR}{dt} + (n+1)(n+2)\frac{\sigma}{\rho R^3}\right] a_n$$

$$+ n(n+1)(n+2)\frac{\nu}{R^2} T(R(t), t)$$

$$+ n(n+1)\frac{1}{R^2}\frac{dR}{dt}\int_{R(t)}^{\infty}[(R/s)^3 - 1](R/s)^n T(s, t)\, ds = 0 .$$

Here the quantity $T(r, t)$ is the toroidal component of the vorticity (Ref. [15], Appendix III) and is the solution of

$$\nu\frac{\partial^2 T}{\partial r^2} - \frac{\partial T}{\partial t} - \frac{\partial}{\partial r}\left[(R/r)^2\frac{dR}{dt}T\right] - n(n+1)\frac{\nu}{r^2}T = 0 , \tag{4}$$

subject to the following condition at the cavity boundary, $r = R(t)$

$$2R^{n-1}\int_{R(t)}^{\infty} s^{-n} T(s, t)\, ds + T(R(t), t) = \tag{5}$$

$$= \frac{2}{n+1}\left[(n+2)\frac{da_n}{dt} - (n-1)\frac{1}{R}\frac{dR}{dt}a_n\right].$$

The initial conditions on the field T depend on the particular problem investigated, and can be assigned with a large degree of arbitrariness. Equation (4) derives from the linearized form of the equation of the vorticity, and Eq. (5) is the expression of the dynamical requirement of vanishing tangential stress at the free surface. Equation (3) then follows on imposing the other dynamical condition that, at the free surface, the discontinuity in the normal stress be balanced by the effect of surface tension.

Upon elimination of $T(R(t), t)$ between Eqs. (3) and (5) one finds

$$\frac{d^2 a_n}{dt^2} + \left[3\frac{1}{R}\frac{dR}{dt} + 2(n+2)(2n+1)\frac{\nu}{R^2}\right]\frac{da_n}{dt} \tag{6}$$

$$+ (n-1)\left[-\frac{1}{R}\frac{d^2 R}{dt^2} + 2(n+2)\frac{\nu}{R^3}\frac{dR}{dt} + (n+1)(n+2)\frac{\sigma}{\rho R^3}\right]a_n$$

$$-2n(n+1)(n+2)\frac{\nu}{R^3}\int_R^\infty (R/s)^n T(s, t)\, ds$$

$$+ n(n+1)\frac{1}{R^2}\frac{dR}{dt}\int_R^\infty [(R/s)^3 - 1](R/s)^n T(s, t)\, ds = 0\,.$$

It can be shown that, if the integrals involving T are expressed in terms of da_n/dt [as is in principle possible by solving Eq. (4) subject to the condition (5)] this equation acquires an integro-differential structure in time. As was already observed in Refs. [8] and [13], this feature arises from the fact that the rate of energy dissipation depends on the instantaneous distribution of vorticity in the liquid, which is determind by the prior motion of the free surface.

Let us consider a situation in which the initial vorticity vanishes (i.e. $T(r, 0)=0$) which would be realized, for instance, for such important cases as purely radial motions or motions started from rest. Equation (5) shows that the irrotational character of the flow cannot persist, but that vorticity will continuously be generated at the bubble surface starting from $t=0$ and will diffuse into the liquid according to Eq. (4). In view of the parabolic nature of this equation we expect that at time t vorticity will be significant only in a spherical shell exterior to the surface $r = R(t)$, the thickness of which can be estimated to be of the order of $(\nu t)^{\frac{1}{2}}$. It is seen therefore that, for small enough times, the two integrals of Eq. (6) are negligible compared with the other terms. In these limiting, but important, situations Eq. (6) simplifies to

$$\frac{d^2 a_n}{dt^2} + \left[3\frac{1}{R}\frac{dR}{dt} + 2(n+2)(2n+1)\frac{\nu}{R^2}\right]\frac{da_n}{dt} \tag{7}$$

$$+ (n-1)\left[(n+1)(n+2)\frac{\sigma}{\rho R^3} - \frac{1}{R}\frac{d^2 R}{dt^2} + 2(n+2)\frac{\nu}{R^3}\frac{dR}{dt}\right]a_n = 0.$$

The viscous contribution that, in contrast with the plane case, appears in the last term should especially be noted here. This equation is the main result of this note and in the following we shall discuss some of its features.

In the first place we observe that, for an inviscid fluid, it becomes

$$\frac{d^2 a_n}{dt^2} + \frac{3}{R}\frac{dR}{dt}\frac{da_n}{dt} \tag{8}$$

$$+ (n-1)\left[-\frac{1}{R}\frac{d^2 R}{dt^2} + (n+1)(n+2)\frac{\sigma}{\rho R^3}\right] a_n = 0 ,$$

a result already obtained by Plesset [11] and Birkhoff [12] (who, however, disregarded the effect of surface tension), and applied by these authors to the study of the stability of the spherical shape for a growing or collapsing bubble [16, 17]. An interesting consequence of Eq. (8) is that for $n = 1$, which corresponds to translation without distortion of the bubble, one readily obtains the integral $R^3(t)\, da_1/dt =$ constant, which is just a statement of the conservation of liquid momentum for a translating sphere of variable radius (see e.g. Ref. [9], Section 92; Ref. [10], Section 11). This result is of course exact, independent of the assumed smallness of a_1. Likewise, we should remark that setting $n = 1$ in the second term of the coefficient of da_n/dt in Eq. (7) gives Levich's well-known expression for the drag force on a translating spherical bubble [18].

For a bubble of constant radius R_0 Eq. (7) reduces to the equation for a damped harmonic oscillator the damping constant and natural frequency of which are

$$\beta_n = (n+2)(2n+1)\,\nu/R_0^2$$

$$\omega_{0,n}^2 = (n-1)(n+1)(n+2)\,\sigma/\rho R_0^3$$

in agreement with the results given by Lamb for a liquid of small viscosity (Ref. [9], p. 641 and p. 475 respectively). It is also interesting to point out that Eq. (2) for the plane case can be obtained from Eq. (7) if R and n are made to tend to infinity in such a way that the wave number $k = n/R$ is held fixed. This definition of the wave number of course is in agreement with the expression for the wavelength λ_n of the surface distortions of order n since, clearly, $\lambda_n = 2\pi R/n$. On performing this limit operation one finds

$$\frac{d^2 a}{dt^2} + 4k\nu^2\frac{da}{dt} - \left(\frac{d^2 R}{dt^2}k - \frac{\sigma}{\rho}k^3\right)a = 0 , \tag{9}$$

from which the similarity with Eq. (2) is apparent. To reconcile the two equations entirely it is sufficient to notice that, according to the physical situation from which Eq. (9) is obtained, $d^2 R/dt^2 > 0$ implies an acceleration directed into the liquid. It will be observed that on performing this limit, the viscous contribution present in the last term of Eq. (7) drops out thus exhibiting its curvature-dependent nature.

In conclusion we would like to examine more closely the conditions under which Eq. (7) represents an acceptable approximation to the complete equation, Eq. (6). For purposes of estimation we make the very crude assumption that $T(r, t) \simeq T(R(t), t)$ for $R \leq r \lesssim R + \delta$ (where $\delta \sim (\nu t)^{\frac{1}{2}}$, approximately), while $T(r, t) \simeq 0$ for $R + \delta \lesssim r$. In this way combination of Eqs. (5) and (6) gives the following first-order correction to Eq. (7)

$$\frac{d^2 a_n}{dt^2} + \left\{3 \frac{1}{R} \frac{dR}{dt} + 2(n+2)(2n+1) \frac{\nu}{R^2} \left[1 - \frac{2n(n+1)}{2n+1} \frac{\delta}{R}\right] - \right.$$

$$\left. - 3n(n+2) \frac{\delta^2}{R^3} \frac{dR}{dt}\right\} \frac{da_n}{dt} + (n-1) \left\{- \frac{1}{R} \frac{d^2 R}{dt^2} + (n+1)(n+2) \frac{\sigma}{\rho R^3} + \right.$$

$$\left. + 2(n+2) \frac{\nu}{R^3} \frac{dR}{dt} \left[1 + 2n \frac{\delta}{R}\right] + 3n \frac{\delta^2}{R^4} \left(\frac{dR}{dt}\right)^2 \right\} a_n \simeq 0 .$$

Upon comparison of this equation with Eq. (7) it is seen that, for n fixed, the following conditions should be satisfied in order that the step leading from (6) to (7) be legitimate

$$\delta \ll R . \tag{10}$$

$$\nu \gg \frac{\delta^2}{R} \left| \frac{dR}{dt} \right| . \tag{11}$$

For ordinary liquids condition (10) is not very restrictive. For instance, bubble phenomena in water occur on time scales of the order of 10^{-3} sec or less; with $\nu = 0.01$ cm^2/sec we then have $(\nu t)^{\frac{1}{2}} \sim 3 \times 10^{-3}$ cm, which will be small compared to R for most cases of practical interest. On the other hand, if the estimate $\delta \sim (\nu t)^{\frac{1}{2}}$ is inserted into (11), one finds $t \ll R \,|\, dR/dt \,|^{-1}$, This condition can be more restrictive than (10), but not so much as to make Eq. (7) practically useless. For example, for a bubble executing forced radial pulsations of angular frequency ω (as would be the case, for instance, for a bubble immersed in a sound field of wavelength much greater than R, (see. e.g. Ref. [19]), we may write

$$R = R_0 (1 + \varepsilon \sin \omega t) ,$$

where ε is the amplitude of the oscillations and R_0 the equilibrium radius of the bubble; condition (11) gives then $t \ll (\omega\varepsilon)^{-1}$. Therefore it is seen that for oscillations of small amplitude ($\varepsilon \ll 1$) Eq. (7) (which would reduce in this case to a damped Mathieu equation) is adequate to describe several features of the motion including the possible development of an instability of the spherical shape. It should be observed that the condition (10) is not uniformly valid in n, in the sense that the error caused by the neglect of the terms of order δ/R increases without bound with increasing n.

As a final point we notice that, while the first integral in Eq. (6) disappears as $\nu \to 0$, the second integral does not. This might seem at variance with the result (8) obtained by Plesset and Birkhoff for an inviscid fluid. The contradiction however is easily resolved by observing that Eq. (8) has been obtained under the assumption or irrotational motion. The possible presence of a non-zero vorticity in the inviscid flow is accounted for by the term in question in Eq. (6). According to Kelvin's theorem (Ref. [9], Section 33; Ref. [10], Section 8) this occurrence is only possible if the vorticity does not vanish at $t = 0$. In addition we also expect that in such a case vorticity is convected with the fluid. Both statements can be readily verified in the present instance since, for $\nu = 0$, Eq. (4) is solved by

$$T(r, t) = r^2 F[r^3 - R^3(t)],$$

where $F[r^3 - R^3(0)] = r^{-2} T(r, 0)$, and the quantity $h = r^3 - R^3(t)$ has the obvious meaning of a Lagrangian variable in the spherical geometry of present concern.

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