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On the geometry of flag manifolds and flag bundles

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Geometria algebrica. — *On the geometry of flag manifolds and flag bundles.* Nota (*) di SAMUEL A. ILORI, presentata dal Corrisp. E. MARTINELLI.

RIASSUNTO. — Si dà una risoluzione parziale del problema di esprimere ogni sottovarietà di Ehresmann di una varietà di bandiere mediante la base di Borel–Hirzebruch dell'anello di coomologia.

§ 1. INTRODUCTION

A flag is a nest of projective subspaces

$$\mathrm{S}: \quad \mathrm{S}_0 \subset \mathrm{S}_1 \subset \cdots \subset \mathrm{S}_{n-1}\,, \quad \dim \mathrm{S}_i = i$$

of complex projective space $\mathrm{P}_n(\mathrm{C})$. The set of all such flags is called the flag manifold of the space $\mathrm{P}_n(\mathrm{C})$ and is denoted by $\mathrm{F} = \mathbf{F}(n+1)$. Cf. p. 324 of [1]). Let V be a non-singular algebraic variety of dimension d, which we suppose to be imbedded in $\mathrm{P}_n(\mathrm{C})$. A tangent flag to V is a $(0, 1, \cdots, d, n)$-flag S with $\mathrm{S}_0 \in \mathrm{V}$ and S_d the tangent $[d]$ to V at S_0. The set of all such flags is called the tangent flag bundle of V and is denoted by V^Δ. (Cf. p. 329 of [1]).

The Borel-Hirzebruch basis of the cohomology ring of $\mathrm{F}(n+1)$ is given on p. 256 of [2] while the Ehresmann basis in terms of permutations $(a_0, \cdots, a_n)$ of $0, 1, \cdots, n$ is given on p. 263 of [2] and in 1.2 of [1]. The purpose of this paper is to give a partial solution to a problem raised by Monk on p. 284 of [2].

The problem is that of finding a formula expressing any given permutation symbol in terms of the Borel-Hirzebruch basis. We shall be using the Ehresmann base as is given on p. 325 in [1] where the integers in an index (permutation symbol) represent codimensions whereas Monk's symbols use actual dimensions. In other words, the index corresponding to a Monk's permutation symbol, $(a_0, \cdots, a_n)$, is $(n - a_0, \cdots, n - a_n)$. If the index $\boldsymbol{k} = (k_0, \cdots, k_q)$ is such that $k_0 < k_1 < \cdots < k_q$, then the problem has been solved and the formula is given in (1.4.3) on [1]. Here we give the formula for an index $\boldsymbol{k} = (k_0, \cdots, k_q)$ where $k_0 > k_1 > \cdots > k_q$.

'Ehresmann' subvarieties of V^Δ are defined in 2.0 of [1] using linear systems of primals on V and indices. We also give the formula expressing an 'Ehresmann' subvariety of V^Δ, with index $\boldsymbol{k} = (k_0, \cdots, k_q)$ where $k_0 > k_1 > \cdots > k_q$, in terms of the basis elements given in 1.5 of [1].

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§ 2. IDENTIFICATION OF BASES ELEMENTS

LEMMA 2.1. *The identification of the bases elements of the cohomology group* $H^2(F(n+1))$ *is given by*

$$\gamma_0 = -[(1);F]^*,$$

$$\gamma_i = [(0,1,\cdots,i-2,i);F]^* - [(0,1,\cdots,i-1,i+1);F]^*, \quad (1 \leq i \leq n-1),$$

and

$$\gamma_n = [(0,1,\cdots,n-2,n);F]^*.$$

Proof. The Ehresmann basis for $H^2(F(n+1))$ is given on p. 265 of [2] and the identification with the Borel-Hirzebruch basis is given in (13.5) of [2].

THEOREM. 2.2. *Let* $\boldsymbol{k} = (a_0,\cdots,a_n)$ *be an index such that* $a_0 > a_1 > \cdots > a_q > 0$ *and such that the Ehresmann subvarieties of* $F(n+1)$, $[(a_0,\cdots,a_n);F]$ *and* $[(a_0,\ldots,a_q);F]$ *are equal. Then in terms of the Borel-Hirzebruch basis elements, the class of the Ehresmann subvariety,* $[(a_0,\cdots,a_q);F]$, *is given by*

$$[(a_0,\cdots,a_q);F]^* = (-1)^m \gamma_0^{a_0}\gamma_1^{a_1}\cdots\gamma_q^{a_q}, \quad \textit{where} \quad m = \sum_{i=0}^{q} a_i.$$

Proof. From Lemma 2.1, to prove the theorem it suffices to prove that

$$[(a_0,\cdots,a_q);F]^* = \prod_{i=0}^{q}\{[(0,1,\cdots,i-1,i+1);F]^* - [(0,\cdots,i-2,i);F]^*\}^{a_i}.$$

The proof is by a series of inductions. When $q=0$ and $a_0=1$, then from Lemma 2.1

$$[(1);F]^* = -\gamma_0.$$

Now as a first inductive hypothesis, assume that

$$[(l);F]^* = \{[(1);F]^*\}^l, \quad \text{for} \quad l < a_0.$$

Then

$$[(a_0);F]^* = [(1);F]^* \cdot [(a_0-1);F]^*,$$

(by the intersection formula on p. 265 of [2]),

$$= [(1);F]^* \cdot \{[(1);F]^*\}^{a_0-1},$$

(by the first inductive hypothesis),

$$= (-1)^{a_0}\gamma_0^{a_0},$$

(by Lemma 2.1).

Next as a second inductive hypothesis, assume that for $p < q$

$$[(a_0, \cdots, a_p); F]^* = \prod_{i=0}^{p} \{[(0, \cdots, i-1, i+1); F]^* -$$
$$- ([0, \cdots, i-2, i); F]^*\}^{a_i}.$$

When $a_q = 1$, we have that

$$[(a_0, \cdots, a_{q-1}); F]^* \cdot \{[(0, \cdots, q-1, q+1); F]^* -$$
$$- [(0, \cdots, q-2, q); F]^*\} = [(a_0, \cdots, a_{q-1}, 1); F]^*,$$

by the intersection formula on p. 265 of [2] since $a_0 > a_1 > \cdots > a_{q-1} > 1$.

Finally assume as a third inductive hypothesis that for $b < a_q$, we have

$$[(a_0, \cdots, a_{q-1}); F]^* \cdot \{[(0, \cdots, q-1, q+1); F]^* -$$
$$- [(0, \cdots, q-2, q); F]^*\}^b = [(a_0, \cdots, a_{q-1}, b); F]^*.$$

In other words,

$$\sum_{i=0}^{b} (-1)^{b-i} \binom{b}{i} \{[(0, \cdots, q-1, q+1); F]^*\}^i \cdot \{[(0, \cdots, q-2, q); F]^*\}^{b-i} \cdot$$
$$\cdot [(a_0, \cdots, a_{q-1}); F]^* = [(a_0, \cdots, a_{q-1}, b); F]^*.$$

Hence

$$\prod_{i=0}^{q} \{[(0, \cdots, i-1, i+1); F]^* - [(0, \cdots, i-2, i); F]^*\}^{a_i} =$$
$$= [(a_0, \cdots, a_{q-1}); F]^* \cdot \{[(0, \cdots, q-1, q+1); F]^* -$$
$$- [(0, \cdots, q-2, q); F]^*\}^{a_q},$$

(by the second inductive hypothesis),

$$= \sum_{i=0}^{a_q} (-1)^{a_q-i} \binom{a_q}{i} \{[(0, \cdots, q-1, q+1); F]^*\}^i \cdot$$
$$\cdot \{[(0, \cdots, q-2, q); F]^*\}^{a_q-i} \cdot [(a_0, \cdots, a_{q-1}); F]^*,$$
$$= \sum_{i=0}^{a_q} (-1)^{a_q-i} \left[\binom{a_q-1}{i} + \binom{a_q-1}{i-1}\right] \{[(0, \cdots, q-1, q+1); F]^*\}^i \cdot$$
$$\cdot \{[(0, \cdots, q-2, q); F]^*\}^{a_q-i} \cdot [(a_0, \cdots, a_{q-1}); F]^*,$$
$$= [(0, \cdots, q-2, q); F]^* \cdot \sum_{i=0}^{a_q-1} (-1)^{a_q-i} \binom{a_q-1}{i} \cdot$$
$$\cdot \{[(0, \cdots, q-1, q+1); F]^*\}^i \cdot \{[(0, \cdots, q-2, q); F]^*\}^{a_q-1-i} \cdot$$
$$\cdot [(a_0, \cdots, a_{q-1}); F]^* + [(0, \cdots, q-1, q+1); F]^* \cdot$$
$$\cdot \sum_{i=1}^{a_q} (-1)^{a_q-i} \binom{a_q-1}{i-1} \{[(0, \cdots, q-1, q+1); F]^*\}^{i-1} \cdot$$
$$\cdot \{[(0, \cdots, q-2, q); F]^*\}^{a_q-i} \cdot [(a_0, \cdots, a_{q-1}); F]^*,$$

$$= - [(0, \cdots, q-2, q); F]^* \cdot \sum_{i=0}^{a_q - 1} (-1)^{a_q - 1 - i} \binom{a_q - 1}{i} \cdot$$
$$\cdot \{[(0, \cdots, q-1, q+1); F]^*\}^i \cdot \{[(0, \cdots, q-2, q); F]^*\}^{a_q - 1 - i} \cdot$$
$$\cdot [(a_0, \cdots, a_{q-1}); F]^* + [(0, \cdots, q-1, q+1); F]^* \cdot$$
$$\cdot \sum_{i=0}^{a_q - 1} (-1)^{a_q - 1 - i} \binom{a_q - 1}{i} \{[(0, \cdots, q-1, q+1); F]^*\}^i \cdot$$
$$\cdot \{[(0, \cdots, q-2, q); F]^*\}^{a_q - 1 - i} \cdot [(a_0, \cdots, a_{q-1}); F]^*,$$
$$= - [(0, \cdots, q-2, q); F]^* \cdot [(a_0, \cdots, a_{q-1}, a_q - 1); F]^* +$$
$$+ [(0, \cdots, q-1, q+1); F]^* \cdot [(a_0, \cdots, a_{q-1}, a_q - 1); F]^*,$$

(by the third inductive hypothesis),

$$= [(a_0, \cdots, a_{q-1}, a_q); F]^*,$$

(by the intersection formula on p. 265 of [2] since $a_0 > a_1 > \cdots > a_{q-1} > a_q - 1$). Thus

$$[(a_0, \cdots, a_q); F]^* = \prod_{i=0}^{q} \{[(0, \cdots, i-1, i+1); F]^* -$$
$$- [(0, \cdots, i-2, i); F]^*\}^{a_i} = (-1)^m \gamma_0^{a_0} \gamma_1^{a_1} \cdots \gamma_q^{a_q}, \quad \text{where } m = \sum_{i=0}^{q} a_i,$$

(by Lemma 2.1), and the theorem is proved.

COROLLARY 2.3. *For any (q, t)-index $\boldsymbol{k} = (a_0, \cdots, a_q)$ where $a_0 > a_1 > \cdots > a_q$, for any non-singular variety V of dimension $\geq q$ and any sufficiently general nest $\mathcal{L}$ of linear systems on V with top dimension $\geq t - 1$, the cohomology class of an 'Ehresmann' subvariety of V^Δ is given by*

$$[(a_0, \cdots, a_q); \mathcal{L} \mid V^\Delta]^* = (\rho^* a)^{a_0} (\rho^* a + \delta_1)^{a_1} \cdots (\rho^* a + \delta_q)^{a_q},$$

where $a \in H^2(V)$ is the cohomology class of the member of $\mathcal{L}_1$.

Proof. By the invariance principle in Theorem 2.5 of [1], the Corollary follows by putting

$$\gamma_0 = - \rho^* a,$$

and $\qquad \gamma_i = - \rho^* a - \delta_i, \qquad 1 \leq i \leq q,$

in the value of $[(a_0, \cdots, a_q); F]^*$ found in the theorem.

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