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Pratap Singh, Awadhesh Kumar Singh

On Kaehlerian conharmonic recurrent and Kaehlerian conharmonic symmetric spaces

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Geometria differenziale. — *On Kaehlerian conharmonic recurrent and Kaehlerian conharmonic symmetric spaces.* Nota di UDAI PRATAP SINGH e AWADHESH KUMAR SINGH, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Vengono studiati certi spazi kaeleriani ricorrenti, detti coarmonici e coarmonici simmetrici.

1. INTRODUCTION

An $n (= 2m)$ dimensional Kaehlerian space K^n is a Riemannian space which admits a structure tensor field φ_i^h satisfying the relations

$$\varphi_j^i \varphi_i^h = -\delta_j^h , \tag{1.1}$$ (1)

$$\varphi_{ij} = -\varphi_{ji} , \qquad (\varphi_{ij} = \varphi_i^a g_{aj}) \tag{1.2}$$

$$\varphi_{i,j}^h = 0 , \tag{1.3}$$

where the comma followed by an index denotes the operator of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemannian curvature tensor, which we denote by R_{ijk}^h, is given by

$$R_{ijk}^h = \partial_i \begin{Bmatrix} h \\ jk \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ ik \end{Bmatrix} + \begin{Bmatrix} h \\ il \end{Bmatrix} \begin{Bmatrix} l \\ jk \end{Bmatrix} - \begin{Bmatrix} h \\ jl \end{Bmatrix} \begin{Bmatrix} l \\ ik \end{Bmatrix} \text{ (2)},$$

whereas the Ricci tensor and the scalar curvature are respectively given by $R_{ij} = R_{aij}^a$ and $R = R_{ij} g^{ij}$.

It is well known that these tensors satisfy the identities ([1]) (3)

$$\varphi_i^a R_a^j = R_i^a \varphi_a^j , \tag{1.4}$$

$$\varphi_i^a R_{aj} = -R_{ia} \varphi_j^a . \tag{1.4$'$}$$

In view of equation (1.1), the relation (1.4) gives

$$\varphi_i^a R_a^b \varphi_b^j = -R_i^j . \tag{1.5}$$

(*) Nella seduta del 12 febbraio 1977.

(1) All Latin indices run over the range from 1 to n.

(2) $\partial_i = \partial/\partial x^i$, where $\{x^i\}$ denotes real local coordinates.

(3) The numbers in square brackets refer to the references at the end of the paper.

Also, multiplying (1.4)′ by g^{ij}, we obtain

$$\varphi_i^a R_a^i = - R_a^j \varphi_j^a ,$$

which implies

$$\varphi_i^a R_a^i = 0 . \tag{1.6}$$

If we define a tensor S_{ij} by

$$S_{ij} = \varphi_i^a R_{aj} , \tag{1.7}$$

we have

$$S_{ij} = - S_{ji} . \tag{1.8}$$

The holomorphically conharmonic curvature tensor T_{ijk}^h and the Bochner curvature tensor K_{ijk}^h are given by ([3])

$$\begin{aligned} T_{ijk}^h = R_{ijk}^h + (R_{ik} \delta_j^h - R_{jk} \delta_i^h + g_{ik} R_j^h - g_{jk} R_i^h + S_{ik} \varphi_j^h - \\ - S_{jk} \varphi_i^h + \varphi_{ik} S_j^h - \varphi_{jk} S_i^h + 2 S_{ij} \varphi_k^h + 2 \varphi_{ij} S_k^h)/(n+4) \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} K_{ijk}^h = R_{ijk}^h + (R_{ik} \delta_j^h - R_{ik} \delta_i^h + g_{ik} R_j^h - g_{jk} R_i^h + S_{ik} \varphi_j^h - \\ - S_{jk} \varphi_i^h + \varphi_{ik} S_j^h - \varphi_{jk} S_i^h + 2 \varphi_k^h S_{ij} + 2 \varphi_{ij} S_k^h)/(n+4) - \\ - R (g_{ik} \delta_j^h - g_{jk} \delta_i^h + \varphi_{ik} \varphi_j^h - \varphi_{jk} \varphi_i^h + 2 \varphi_{ij} \varphi_k^h)/[(n+2)(n+4)] \end{aligned} \tag{1.10}$$

respectively.

The equation (1.10), in view of (1.9), may be expressed as

$$\begin{aligned} K_{ijk}^h = T_{ijk}^h - R (g_{ik} \delta_j^h - g_{jk} \delta_i^h + \varphi_{ik} \varphi_j^h - \\ - \varphi_{jk} \varphi_i^h + 2 \varphi_{ij} \varphi_k^h)/[(n+2)(n+4)] . \end{aligned} \tag{1.11}$$

We shall use the following

DEFINITION (1.1). A Kaehler space satisfying ([2])

$$R_{ijk,a}^h = \lambda_a R_{ijk}^h \tag{1.12}$$

for some non-zero vector field λ_a, will be called a Kaehlerian recurrent space. The space K^n is called Kaehlerian Ricci—recurrent if it satisfies the relation

$$R_{ij,a} = \lambda_a R_{ij} ; \tag{1.13}$$

then, multiplying the above equation by g^{ij} and using the fact that $g^{ij}{}_{,a} = 0$, we get

$$R_{,a} = \lambda_a R . \tag{1.13}$$

REMARK (1.1). From (1.13) it follows that every Kaehlerian recurrent space is Kaehlerian Ricci-recurrent; but the converse is not necessarily true.

DEFINITION (1.2). A Kaehler space is called Kaehlerian symmetric in the sense of Cartan if it satisfies ([2])

$$(1.14) \qquad R^h_{ijk,a} = 0 \qquad \text{or equivalently } R_{ijkl,a} = 0 .$$

Obviously a Kaehlerian symmetric space is Kaehlerian Ricci-symmetric, i.e.

$$(1.14)' \qquad R_{ij,a} = 0 .$$

DEFINITION (1.3). A Kaehler space in which the Bochner curvature tensor K^h_{ijk} satisfies the relation

$$(1.15) \qquad K^h_{ijk,a} = \lambda_a K^h_{ijk}$$

for some non-zero vector λ_a, will be called a Kaehler space with recurrent Bochner curvature tensor, or Kaehlerian Bochner recurrent space ([2]).

In the present paper, we give several theorems in Kaehlerian conharmonic recurrent and Kaehlerian conharmonic symmetric spaces.

2. KAEHLERIAN CONHARMONIC RECURRENT SPACE

DEFINITION (2.1). A Kahler space satisfying the relation

$$(2.1) \qquad T^h_{ijk,a} = \lambda_a T^h_{ijk}$$

for some non-zero recurrence vector λ_a, will be called a Kaehlerian conharmonic recurrent space.

THEOREM (2.1). *Every Kaehlerian recurrent space is Kaehlerian conharmonic recurrent.*

Proof. A Kaehlerian recurrent space is characterized by the equation (1.12), which yields (1.13). By differentiating (1.9) covariantly with respect to x^a and using equations (1.7) and (1.13), we get

$$T^h_{ijk,a} = \lambda_a T^h_{ijk} ,$$

which shows that the space is Kaehlerian conharmonic recurrent.

THEOREM (2.2). *Every Kaehlerian conharmonic recurrent space is a Kaehler space with recurrent Bochner curvature tensor.*

Proof. Let the space be Kaehlerian conharmonic recurrent. Equation (2.1), in view of (1.9) gives

$$(2.2) \qquad \begin{aligned} & R^h_{ijk,a} + (\delta^h_j R_{ik,a} - \delta^h_i R_{jk,a} + g_{ik} R^h_{j,a} - g_{jk} R^h_{i,a} + \varphi^h_j S_{ik,a} - \\ & - \varphi^h_i S_{jk,a} + \varphi_{ik} S^h_{j,a} - \varphi_{jk} S^h_{i,a} + 2\varphi^h_k S_{ij,a} + 2\varphi_{ij} S^h_{k,a})/(n+4) = \\ & = \lambda_a [R^h_{ijk} + (R_{ik}\delta^h_j - R_{jk}\delta^h_i + g_{ik} R^h_j - g_{ik} R^h_i + S_{ik}\varphi^h_j - \\ & - S_{jk}\varphi^h_i + \varphi_{ik} S^h_j - \varphi_{jk} S^h_i + 2 S_{ij}\varphi^h_k + 2\varphi_{ij} S^h_k)/(n+4)] . \end{aligned}$$

Multiplying the above equation by g^{jk} and making some simplifications with the help of equations (1.4), (1.5), (1.6) and (1.7) and using also the identity $\varphi_{ij}\, g^{ij} = 0$, we get

$$\delta_i^h\,(\mathrm{R}_{,a} - \lambda_a\,\mathrm{R}) = 0\,, \tag{2.3}$$

which gives

$$\mathrm{R}_{,a} - \lambda_a\,\mathrm{R} = 0\,. \tag{2.4}$$

Differentiating (1.11) covariantly with respect to x^a, we obtain

$$\begin{aligned}\mathrm{K}^h_{ijk,a} = \mathrm{T}^h_{ijk,a} - \mathrm{R}_{,a}\,(g_{ik}\,\delta_j^h - g_{jk}\,\delta_i^h + \varphi_{ik}\,\varphi_j^h - \\ - \varphi_{jk}\,\varphi_i^h + 2\,\varphi_{ij}\,\varphi_k^h)/[(n+2)\,(n+4)]\,.\end{aligned} \tag{2.5}$$

Multiplying (1.9) by λ_a and subtracting from (2.5), we have

$$\begin{aligned}\mathrm{K}^h_{ijk,a} - \lambda_a\,\mathrm{K}^h_{ijk} = \mathrm{T}^h_{ijk,a} - \lambda_a\,\mathrm{T}^h_{ijk} - (\mathrm{R}_{,a} - \lambda_a\,\mathrm{R})\,(g_{ik}\,\delta_j^h - \\ - g_{jk}\,\delta_i^h + \varphi_{ik}\,\varphi_j^h - \varphi_{jk}\,\varphi_i^h + 2\,\varphi_{ij}\,\varphi_k^h)/[(n+2)\,(n+4)]\,.\end{aligned} \tag{2.6}$$

Using equations (2.1) and (2.4) in (2.6), we obtain

$$\mathrm{K}^h_{ijk,a} - \lambda_a\,\mathrm{K}^h_{ijk} = 0\,,$$

which shows that the space is a Kaehler space with recurrent Bochner curvature tensor.

THEOREM (2.3). *The necessary and sufficient conditions for a Kaehler space to be Kaehlerian conharmonic recurrent are that the space be of recurrent Bochner curvature and that*

$$\mathrm{R}_{,a} - \lambda_a\,\mathrm{R} = 0$$

holds.

Proof. The necessary part has been proved in Theorem (2.2). For the sufficient condition, suppose that the space be of recurrent Bochner curvature and that

$$\mathrm{R}_{,a} - \lambda_a\,\mathrm{R} = 0$$

holds. The equation (1.11) yields

$$\begin{aligned}\mathrm{K}^h_{ijk,a} - \lambda_a\,\mathrm{K}^h_{ijk} = \mathrm{T}^h_{ijk,a} - \lambda_a\,\mathrm{T}^h_{ijk} - (\mathrm{R}_{,a} - \lambda_a\,\mathrm{R})\,(g_{ik}\,\delta_j^h - g_{jk}\,\delta_i^h + \\ + \varphi_{ik}\,\varphi_j^h - \varphi_{jk}\,\varphi_i^h + 2\,\varphi_{ij}\,\varphi_k^h)/[(n+2)\,(n+4)]\,.\end{aligned}$$

This equation, after using the above mentioned conditions, gives

$$\mathrm{T}^h_{ijk,a} - \lambda_a\,\mathrm{T}^h_{ijk} = 0\,,$$

which shows that the space is Kaehlerian conharmonic recurrent. This completes the proof.

THEOREM (2.4). *A Kaehlerian conharmonic recurrent space will be Kaehlerian recurrent provided that it is Kaehlerian Ricci-recurrent.*

Proof. Differentiating (1.9) covariantly with respect to x^a, we obtain

$$(2.7)\qquad T^h_{ijk,a} = R^h_{ijk,a} + (\delta^h_j R_{ik,a} - \delta^h_i R_{jk,a} + g_{ik} R^h_{j,a} - g_{jk} R^h_{i,a} + \varphi^h_j S_{ik,a} - \\ - \varphi^h_i S_{jk,a} + \varphi_{ik} S^h_{j,a} - \varphi_{jk} S^h_{i,a} + 2\,\varphi^h_k S_{ij,a} + 2\,\varphi_{ij} S^h_{k,a})/(n+4)\,.$$

Multiplying equation (1.9) by λ_a and subtracting from (2.7), we have

$$(2.8)\qquad T^h_{ijk,a} - \lambda_a T^h_{ijk} = R^h_{ijk,a} \lambda_a R^h_{ijk} + [\delta^h_j (R_{ik,a} - \lambda_a R_{ik}) - \\ - \delta^h_i (R_{jk,a} - \lambda_a R_{jk}) + g_{ik} (R^h_{j,a} - \lambda_a R^h_j) - g_{jk} (R^h_{i,a} - \lambda_a R^h_i) + \\ + \varphi^h_j (S_{ik,a} - \lambda_a S_{ik}) - \varphi^h_i (S_{jk,a} - \lambda_a S_{jk}) + \varphi_{ik} (S^h_{j,a} - \lambda_a S^h_j) - \\ - \varphi_{jk} (S^h_{i,a} - \lambda_a S^h_i) + 2\,\varphi^h_k (S_{ij,a} - \lambda_a S_{ij}) + 2\,\varphi_{ij} (S^h_{k,a} - \lambda_a S^h_k)]/(n+4).$$

Let the space be Kaehlerian Ricci-recurrent. Then equation (2.8) yields

$$(2.9)\qquad T^h_{ijk,a} - \lambda_a T^h_{ijk} = R^h_{ijk,a} - \lambda_a R^h_{ijk}\,,$$

which shows that the Kaehlerian conharmonic recurrent space is Kaehlerian recurrent.

THEOREM (2.5). *The necessary and sufficient condition that a Kaehler space is Kaehlerian Ricci-recurrent, is that*

$$T^h_{ijk,a} - \lambda_a T^h_{ijk} = R^h_{ijk,a} - \lambda_a R^h_{ijk}\,.$$

Proof. Let the space be Kaehlerian Ricci-recurrent, then the relation (1.13) is satisfied and so the equation (2.8) reduces to

$$T^h_{ijk,a} - \lambda_a T^h_{ijk} = R^h_{ijk,a} - \lambda_a R^h_{ijk}\,.$$

Conversely, if in a Kaehler space the above equation is satisfied, then equation (2.8) yields

$$(2.10)\qquad \delta^h_j (R_{ik,a} - \lambda_a R_{ik}) - \delta^h_i (R_{jk,a} - \lambda_a R_{jk}) + g_{ik} (R^h_{j,a} - \lambda_a R^h_j) - \\ - g_{jk} (R^h_{i,a} - \lambda_a R^h_i) + \varphi^h_j (S_{ik,a} - \lambda_a S_{ik}) - \varphi^h_i (S_{jk,a} - \lambda_a S_{jk}) + \\ + \varphi_{ik} (S^h_{j,a} - \lambda_a S^h_j) - \varphi_{jk} (S^h_{i,a} - \lambda_a S^h_i) + 2\,\varphi^h_k (S_{ij,a} - \lambda_a S_{ij}) + \\ + 2\,\varphi_{ij} (S^h_{k,a} - \lambda_a S^h_k) = 0\,.$$

Simplifying with the help of equations (1.4), (1.5), (1.6) and (1.7), and also using the identity $\varphi_{ij}\, g^{ij} = 0$, we get

$$(2.11)\qquad (n+2)(R_{ik,a} - \lambda_a R_{ik}) = 0\,,$$

which gives

$$R_{ik,a} - \lambda_a R_{ik} = 0 .$$

This shows that the space is Kaehlerian Ricci-recurrent, which completes the proof.

The following theorem is immediate from (2.9):

THEOREM (2.6). *The necessary and sufficient condition for a Kaehlerian Ricci-recurrent space to be Kaehlerian recurrent is that the space be Kaehlerian conharmonic recurrent.*

3. KAEHLERIAN CONHARMONIC SYMMETRIC SPACE

DEFINITION (3.1). A Kaehler space satisfying the relation

$$T^h_{ijk,a} = 0 , \qquad \text{or equivalently} \quad T_{ijkl,a} = 0 , \tag{3.1}$$

will be called a Kaehlerian conharmonic symmetric space.

THEOREM (3.1). *Every Kaehlerian symmetric space is a Kaehlerian conharmonic symmetric space.*

Proof. If the space is Kaehlerian symmetric, then the relations (1.14) and (1.14)′ are satisfied and so the equation (2.7) gives

$$T^h_{ijk,a} = 0 ,$$

which shows that the space is Kaehlerian conharmonic symmetric.

THEOREM (3.2). *The necessary and sufficient condition that a Kaehlerian conharmonic symmetric space be Kaehlerian Ricci recurrent, with recurrence vector λ_a, is that*

$$R^h_{ijk,a} + \lambda_a (T^h_{ijk} - R^h_{ijk}) = 0 .$$

Proof. Since the space is Kaehlerian conharmonic symmetric, the equation (2.8) takes the form

$$\begin{aligned} R^h_{ijk,a} - \lambda_a R^h_{ijk} + \lambda_a T^h_{ijk} + [\delta^h_j (R_{ik,a} - \lambda_a R_{ik}) - \delta^h_i (R_{jk,a} - \lambda_a R_{jk}) + \\ + g_{ik} (R^h_{j,a} - \lambda_a R^h_j) - g_{jk} (R^h_{i,a} - \lambda_a R^h_i) + \varphi^h_j (S_{ik,a} - \lambda_a S_{ik}) - \\ - \varphi^h_i (S_{jk,a} - \lambda_a S_{jk}) + \varphi_{ik} (S^h_{j,a} - \lambda_a S^h_j) - \varphi_{jk} (S^h_{i,a} - \lambda_a S^h_i) + \\ + 2 \varphi^h_k (S_{ij,a} - \lambda_a S_{ij}) + 2 \varphi_{ij} (S^h_{k,a} - \lambda_a S^h_k)]/(n+4) = 0 . \end{aligned} \tag{3.2}$$

If the space is Kaehlerian Ricci-recurrent, the above equation reduces to

$$R^h_{ijk,a} - \lambda_a R^h_{ijk} + \lambda_a T^h_{ijk} = 0 . \tag{3.3}$$

Conversely, if this equation is satisfied, then proceeding as in Theorem (2.5) it can be seen that the space is Kaehlerian Ricci-recurrent.

THEOREM (3.3). *In a Kaehlerian conharmonic symmetric space the scalar curvature is constant.*

Proof. From equations (1.9) and (3.1), we obtain

$$(3.3)' \qquad R^h_{ijk,a} = -(\delta^h_j R_{ik,a} - \delta^h_i R_{jk,a} + g_{ik} R^h_{j,a} - g_{jk} R^h_{i,a} + \varphi^h_j S_{ik,a} - \\ - \varphi^h_i S_{jk,a} + \varphi_{ik} S^h_{j,a} - \varphi_{jk} S^h_{i,a} + 2\varphi^h_k S_{ij,a} + 2\varphi_{ij} S^h_{k,a})/(n+4).$$

Multiplying this equation by g^{jk}, making use of equations (1.4), (1.5), (1.6) and (1.7), and also using the identity $\varphi_{jk} g^{jk} = 0$, we get

$$(3.4) \qquad \delta^h_i R_{,a} = 0 ,$$

which gives

$$(3.5) \qquad R_{,a} = 0 , \quad \text{i.e. } R \text{ is a constant.}$$

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