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# Rendiconti

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## A comparison theorem for general n-th order functional differential nonlinear equations with deviating arguments

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**Equazioni differenziali ordinarie.** — *A comparison theorem for general n-th order functional differential nonlinear equations with deviating arguments* (*). Nota di LU-SAN CHEN e CHEH-CHIH YEH, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori estendono ad un'equazione differenziale di ordine $n$ con argomenti ritardati alcuni risultati di *A. G. Kartsatos* e *H. Onose*.

## 1. INTRODUCTION

The results of this paper are inspired by two recent papers of Kartsatos-Onose [5] and Onose [7]. The method of the authors can be used to prove two comparison theorems for more general $n$-th order functional differential nonlinear equations with deviating arguments

$$L_n x(t) + H_j(t, X[g(t)]) = Q(t), \qquad j = 1, 2. \tag{1}$$

It is assumed throughout this paper that

(α) $r_k(t) \in C[R_+ \equiv [0, \infty), R_+ \setminus \{0\}]$, $\int^{\infty} r_k(t)\, dt = \infty$, $k = 1, 2, \cdots, n$;

(β) $H_j(t, Y) \in C[R_+ \times R^m, R \equiv (-\infty, \infty)]$,
$Y \equiv (y_1, \cdots, y_m) > 0$ (i.e. $y_i > 0$ *for* $i = 1, 2, \cdots, m) \rightarrow$
$\rightarrow H_j(t, Y) > 0$,
$Y \equiv (y_1, \cdots, y_m) < 0$ (i.e. $y_i < 0$ *for* $i = 1, 2, \cdots, m) \rightarrow$
$\rightarrow H_j(t, Y) < 0$;

*and* $H_j(t, Y)$ *is nondecreasing with respect to* $Y$, i.e.

$$X \equiv (x_1, \cdots, x_m) \geq Y \equiv (y_1, \cdots, y_m) \text{ (i.e. } x_i \geq y_i, i = 1, 2, \cdots, m) \rightarrow$$
$$\rightarrow H_j(t, X) \geq H_j(t, Y) \quad \textit{for } j = 1, 2.$$

(γ) $Q(t) \in C[R_+, R]$ *and there is an oscillatory function* $f(t) \in C^n[R_+, R]$ *such that*

$$L_n f(t) \equiv Q(t) \quad \textit{and} \quad \lim_{t \to \infty} L_\varkappa f(t) = 0, \varkappa = 0, 1, \cdots, n-1.$$

(δ) $g_i(t) \in C[R_+, R]$, $\lim_{t \to \infty} g_i(t) = \infty$, $\qquad i = 1, 2, \cdots, m$.

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A nontrivial solution of (1) which exists on $[c, \infty)$, for some fixed $c \geq 0$, is called *oscillatory* if it has arbitrary large zeros. Otherwise, it is said to be *nonoscillatory*. Equation (1) itself is called oscillatory if all solutions of (1) are oscillatory.

The reader is referred to Chen-Yeh [1-3], Kartsatos [4], Onose [7] and Kartsatos-Onose [5].

In order to obtain our main results, we need the following Kiguradze's Lemma [6] which has been generalized by Chen-Yeh [1].

LEMMA. *Suppose $u(t) > 0$ and $L_n u(t) \leq 0$ for $t \geq c$. Then there exsits an integer $k$ with $0 \leq k \leq n-1$ and $n+k$ odd such that for $t$ large enough*

$$\begin{cases} L_\varkappa u(t) > 0, & \varkappa = 0, 1, \cdots, k, \\ (-1)^{k+\varkappa} L_\varkappa u(t) \geq 0, & \varkappa = k+1, \cdots, n. \end{cases} \tag{2}$$

## 2. COMPARISON THEOREM

THEOREM 1. *Consider the equation.*

$$L_n x(t) + H_1(t, X[g(t)]) = Q(t) \tag{3}$$

*and the inequality*

$$L_n x(t) + H_2(t, X[g(t)]) \leq Q(t) \tag{4}$$

*where*

$$H_1(t, X) \leq H_2(t, X), \qquad t \in R_+, \quad X \geq 0,$$

$$H_1(t, X) \geq H_2(t, X), \qquad t \in R_+, \quad X \leq 0.$$

*If equation* (3) *is oscillatory for $n$ even or a solution $x(t)$ of* (3) *is oscillatory or* $\liminf_{t\to\infty} x(t) = 0$ *for $n$ odd, then the same fact holds for equation* (4).

*Proof*. Let $x(t)$ be a nonoscillatory solution of (4) and $\liminf_{t\to\infty} |x(t)| > 0$ for $n$ odd. Without loss of generality, we may assume that $x(t) > 0$ for $t \geq c$. It follows from ($\delta$) that there exists a $T \geq c$ such that for $t \geq T$.

$$x[g_i(t)] > 0, \qquad i = 1, 2, \cdots, m.$$

Then the function $y(t) \equiv x(t) - f(t)$ is an eventually positive solution of the equation

$$L_n y(t) + H_2(t, Y[g(t)] + F[g(t)]) \leq 0 \tag{5}$$

where $Y[g(t)] = (y[g_1(t)], \cdots, y[g_m(t)])$ and $F[g(t)] = (f[g_1(t)], \cdots, f[g_m(t)])$. In fact, $Y[g(t)] + F[g(t)] > 0$ for $t \geq T$, which implies $L_n y(t) < 0$ for $t \geq T$. Consequently, $y(t)$ has to be eventually of constant sign. If $y(t) < 0$ for all large $t$, then $f(t) > -y(t) > 0$ for all large $t$, a contradiction to the oscillatory character of $f(t)$. Hence

$$y(t) > 0 \tag{6}$$

eventually. It follows from Kiguradze's Lemma that there exists $k$ with $0 \leq k \leq n-1$ and $n+k$ odd such that (2) holds. In particular $y(t) > 0$, $y'(t) > 0$ if $n$ is even or odd, or possibly for $n$ odd, $y(t) > 0$, $y'(t) < 0$ for every $t \geq T$. Let now T be so large that we also have $|f(t)| < M \leq y(T)$, where M is a positive constant.

Integrating (5) $n$-times, we have for $t \geq T$

$$y(t) \geq M + \int_T^t r_1(u_{n-1}) \int_T^{u_{n-1}} r_2(u_{n-2}) \cdots \int_T^{u_{n-k+1}} r_k(u_{n-k}) \int_{u_{n-k}}^{\infty} r_{k+1}(u_{n-k-1}) \tag{7}$$

$$\cdots \int_{u_1}^{\infty} H_2(u, Y[g(s)] + F[g(s)]) \, ds \, du_1 \cdots du_{n-1}$$

$$\equiv M + \varnothing(t, H_2(t, Y[g(t)] + F[g(t)]))$$

$$\geq M + \varnothing(t, H_1(t, Y[g(t)] + F[g(t)]))$$

where $M = y(T)$ if $y'(t) > 0$ and $M = \dfrac{y(T)}{2}$ if $y'(t) < 0$.

Now define

$$z_0(t) = y(t) \qquad \text{for} \quad t \geq c,$$

$$z_{n+1}(t) = \begin{cases} M + \varnothing(t, H_1(t, Z_n[g(t)] + F[g(t)])) & \text{for} \quad t \geq T \\ M \quad \text{for} \quad c \leq t \leq T. \end{cases}$$

Then from (7) we obtain by induction that for $t \geq T$ and $n = 0, 1, \cdots$.

$$0 < z_n(t) \leq y(t),$$

$$M \leq z_{n+1}(t) \leq z_n(t).$$

Consequently, letting $\lim_{t \to \infty} z_n(t) = z(t)$ for $t \geq T$ and applying Lebesgue's theorem on monotone convergence we get for $t \geq T$

$$z(t) = M + \varnothing(t, H_1(t, Z[g(t)] + F[g(t)])). \tag{8}$$

Differentiating (8) $n$-times, we have

$$L_n z(t) + H_1(t, Z[g(t)] + F[g(t)]) = 0.$$

Letting $r(t) = z(t) + f(t)$, we get for $t \geq T$

$$\text{(9)} \qquad L_n r(t) + H_1(t, R[g(t)]) = Q(t).$$

Since $z(t) + f(t) > M + f(t) > 0$, (9) has an eventually positive solution, or for $n$ odd, $\liminf_{t\to\infty} r(t) > 0$, a contradiction.

REMARK 1. Taking $r_\varkappa(t) = 1$, $\varkappa = 1, 2, \cdots, n$, $H_1 = H_2$ and $Q(t) = 0$, then Onose's result [7, Theorem 1] is a special case of our Theorem 1.

REMARK 2. Taking $r_\varkappa(t) = 1$, $\varkappa = 1, 2, \cdots, n$, $m = 1$, and if the equality of (4) holds, then Kartsatos-Onose's result [5] is a special case of our Theorem 1.

The following extends a result due to Onose [7, Theorem 4].

THEOREM 2. *Let the conditions of Theorem 1 hold. Suppose there exist constants $q_1, q_2$ and sequences $\{t_s'\}, \{t_s''\}$ such that*

$$\lim_{s\to\infty} t_s' = \lim_{s\to\infty} t_s'' = \infty \;,\; f(t_s') = q_1, f(t_s'') = q_2 \quad \text{and} \quad q_1 \leq f(t) \leq q_2 .$$

*For $n$ odd if every solution $x(t)$ of* (3) *is oscillatory or* $\liminf_{t\to\infty} x(t) = 0$, *then every solution of* (4) *is oscillatory or* $\lim_{t\to\infty} [x(t) - f(t)] = -q_1$ *(or* $-q_2$*), while for $n$ even, if* (3) *is oscillatory, then* (4) *is oscillatory.*

*Proof.* The same procedure of the proof of Theorem 1 is used. Suppose $x(t)$ is a nonoscillatory positive solution of (4) for $t \geq c$. Put $y(t) = x(t) - f(t)$. Then we see that our proof proceeds until (6). Put $z(t) = y(t) + q_1$. Since $y'(t)$ is monotone, we have

$$\lim_{t\to\infty} z(t) = c \; (-\infty \leq c \leq \infty).$$

If $c < 0$, then we have $y(t) + q_1 < 0$ for $t$ large enough; this leads to a contradiction to the fact that

$$y(t_s') + q_1 = y(t_s') + f(t_s') = x(t_s') > 0 .$$

If $c = 0$, then we have that $\lim_{t\to\infty} [x(t) - f(t)] = -q_1$. If $c > 0$, then we have

$$0 < z(t) = y(t) + q_1 \leq y(t) + f(t).$$

It follows from (5) and ($\beta$) that for $t \geq T$

$$\text{(10)} \qquad L_n z(t) + H_1(t, Z[g(t)]) \leq L_n z(t) + H_2(t, Z[g(t)]) \leq 0.$$

Hence we consider the following equation

$$\text{(11)} \qquad L_n z(t) + H_1(t, Z[g(t)]) = 0 \quad \text{for } t \geq T.$$

From (10), (11) and Theorem 1, we complete our proof.

## References


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$$\frac{d^m u}{dt^m} + a(t)\,|u|^n \operatorname{sign} u = 0.$$

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