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On the stability of motion and perturbation of Lyapunov functions

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Equazioni differenziali ordinarie. — *On the stability of motion and perturbation of Lyapunov functions.* Nota di OLUSOLA AKINYELE, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore con l'impiego di funzioni di Lyapunov dà criteri sufficienti, abbastanza generali, per la equistabilità, l'equiasintotica stabilità, la forte equistabilità delle soluzioni di un sistema differenziale.

1. INTRODUCTION

We shall consider the system of differential equations:

$$\frac{dx}{dt} = f(t, x) \quad , \quad x(t_0) = x_0 \tag{1}$$

where $f \in C(R \times R^n, R^n)$. Here R denotes the real line, R^n the Euclidean space and $C(R \times R^n, R^n)$ the class of continuous functions from $R \times R^n$ to R^n. For any $\rho > 0$, let $S(\rho) = \{x \in R^n : \|x\| < \rho\}$, $\|\cdot\|$ being any convenient norm in R^n. We here consider the stability results which allow the initial time t_0 the freedom of taking any value in the interval $(-\infty, \infty)$, although the conditions imposed on the Lyapunov functions used in our study are only for $t \geq 0$.

The idea of perturbing Lyapunov functions was introduced in [1], to discuss non-uniform properties of solutions of systems of differential equations under weaker assumptions. The results of [1], thus clearly show how many advantages there are in employing the perturbations of Lyapunov functions technique in studying the boundedness and stability criteria of systems of ordinary differential equations. In fact, the equiboundedness property was proved without assuming conditions everywhere in R^n, as in the case of uniform boundedness. The corresponding situation relative to equistability was also discussed. In this paper, we use the technique of perturbing Lyapunov functions to discuss perfect equistability ad perfect equi-asymptotic stability of the differential system [1], under weaker assumptions. If $t \in [0, \infty)$, then we obtain the sufficient conditions for the strong equistability of system (1) under weaker assumptions. Known results [2] on these types of stability criteria require that certain assumptions hold everywhere in $S(\rho)$, but we shall relax this requirement by perturbing the Lyapunov functions. Our results thus improve considerably the perfect equistability, the perfect equi-asymptotic and the strong equistability results of [2].

(*) Nella seduta del 12 febbraio 1977.

2. MAIN RESULTS

We give a number of definitions:

DEFINITION 2.1.

(i) The trivial solution $x = 0$ of (1) is perfectly equistable if for $\varepsilon > 0$, $t_0 \in (-\infty, \infty)$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ which is continuous in t_0 for each $\varepsilon > 0$ such that the inequality

$$\| x_0 \| \leq \delta$$

implies

$$\| x(t, t_0, x_0) \| < \varepsilon \qquad t \geq t_0 .$$

(ii) The trivial solution $x = 0$ of (1) is perfectly equi-asymptotically stable if (i) holds and there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \varepsilon)$ such that $t \geq t_0 + T$ and $\| x_0 \| < \delta_0$ implies

$$\| x(t, t_0, x_0) \| < \varepsilon .$$

DEFINITION 2.2.

The trivial solution $x = 0$ of (1) is said to be strongly equistable if for any $\varepsilon > 0, t_0 \in [0, \infty)$ and any compact interval $[t_0, t_1]$, there exist $\eta = \eta(\varepsilon) > 0$ and a positive function $\delta = \delta(t_0, \varepsilon)$ which is continuous in t_0 for each ε such that if $\| x_0 \| \leq \delta$,

$$\| x(t, t_0, x_0, \eta) \| < \varepsilon \qquad t \in [t_0, t_1] ,$$

where $x(t, t_0, x_0, \eta)$ is an η-approximate solution of (1) on $[t_0, t_1]$. If δ is independent of t_0, the trivial solution is said to be strongly uniform stable.

Remark. The notions of strong equistability and strong uniform stability with respect to the scalar differential equation

$$\frac{du}{dt} = g(t, u) \qquad u(t_0) = u_0 \geq 0 \qquad t \in [0, \infty)$$

are defined as in Definition 2.2.

THEOREM 2.3. *Assume that*

(i) $V_1 \in C(R^+ \times S(\rho), R^+)$, $V_1(t, x)$ *is locally Lipt. in* x *for a constant* $L = L(\rho) > 0$, $V_1(t, 0) = 0$ *and*

$$D^+ V_1(t, x) \leq g_1(t, V_1(t, x)) ,$$

$(t, x) \in R^+ \times S(\rho)$, *where* $g_1 \in (C(R^+ \times R^+, R)$ *and* $g_1(t, 0) = 0$.

(ii) *For every* $\tau > 0$, *there exists a*

$$V_{2,\tau} \in C(R^+ \times S(\rho) \cap S^c(\tau), R^+), S^c(\tau)$$

is the complement of $S(\tau)$, $V_{2,\tau}$ *is locally Lip in* x,

$$b(\|x\|) \leq V_{2,\tau}(t, x) \leq a(\|x\|), \ (t, x) \in R^+ \times S(\rho) \cap S^c(\tau),$$

where $a, b, \in C[(0, \rho), R^+]$, $a(u)$, $b(u)$ *increasing in* u *and* $a(u) \to 0$ *as* $u \to 0$ *and*

$$D^+ V_1(t, x) + D^+ V_{2,\tau}(t, x) \leq g_2(t, V_1(t, x) + V_{2,\tau}(t, x))$$

for

$$(t, x) \in (R^+ \times S(\rho) \cap S^c(\tau))$$

where

$$g_2 \in C(R^+ \times R^+, R), \qquad g_2(t, 0) \equiv 0;$$

(iii) $f \in C([-\infty, \infty] \times S(\rho), R^n)$, $f(t, 0) = 0$ *and* $f(t, x)$ *is almost periodic in* t *uniformly with respect to* $x \in S$, S *being any compact set in* $S(\rho)$;

(iv) *The trivial solution* $u = 0$ *is strongly equistable with respect to the scalar differential equation.*

$$u^1 = g_1(t, u) \quad , \quad u(t_0) = u_0 \geq 0 \tag{2}$$

and strongly uniformly stable with respect to the scalar differential equation

$$w^1 = g_2(t, w) \quad , \quad w(t_0) = w_0 \geq 0 . \tag{3}$$

Then the trivial solution of $x = 0$ *of the system* (1) *is perfectly equistable.*

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in (-\infty, \infty)$ be given. Since the trivial solution is strongly uniform stable with respect to (3), given $b(\varepsilon) > 0$ and $\tau_0 \in R^+$, and any compact interval $K = [\tau_0, t_1^*]$, $\exists = \eta(\varepsilon) > 0$ and a positive $\delta_0 = \delta_0(\varepsilon) > 0$ such that

$$w_0 < \delta_0, \quad \text{implies} \quad w(t, \tau_0, w_0, \eta) < b(\varepsilon), \ t \in [\tau_0, t_1^*], \tag{4}$$

where $w(t, \tau_0, w_0, \eta)$ is any solution of

$$w^1 = g_2(t, w) + \eta \quad , \quad w(\tau_0) = w_0 \geq 0 . \tag{5}$$

Because of the hypothesis on $a(u)$, there exists $\delta_2 = \delta_2(\varepsilon) > 0$ such that

$$a(\delta_2) < \frac{\delta_0}{2} . \tag{6}$$

Since the trivial solution of (2) is strongly equistable, given $\frac{\delta_0}{2} > 0$ and $\tau_0 \in R^+$ and any compact interval $K_1 = [\tau_0, t_2^*]$, $\exists \eta_1 = \eta_1(\varepsilon) > 0$ and $\delta^* = \delta^*(\tau_0, \varepsilon)$ such that

$$u_0 < \delta^* \quad \text{implies} \quad u(t, \tau_0, u_0, \eta_1) < \frac{\delta_0}{2}, \ t \in [\tau_0, t_2^*], \tag{7}$$

where $u(t, \tau_0, u_0, \eta_1)$ is any solution of

$$u^1 = g_1(t, u) + \eta_1 \quad , \quad u(\tau_0) = u_0 \geq 0 . \tag{8}$$

Let $x(t, t_0, x_0)$ be a solution of (1) with $x(t_0) = x_0$ and choose $u_0 = V_1(t_0, x_0)$. Since $V_1(t, x)$ is continuous and $V_1(t, 0) = 0$ there exists $\delta_1 = \delta_1(t_0, \varepsilon)$ such that

$$\| x_0 \| < \delta_1 \quad \text{and} \quad V_1(t_0, x_0) < \delta^* \quad \text{hold} \tag{9}$$

simultaneously. Set $\delta = \min\{\delta_1, \delta_2\}$; we claim that

$$\| x_0 \| < \delta \quad \text{implies} \quad \| x(t, t_0, x_0) \| < \varepsilon \quad \text{for } t \geq t_0 ,$$

with $t_0 \in (-\infty, \infty)$. Suppose not, then there exists a solution $x(t, t_0, x_0)$ of (1) with $\| x_0 \| < \delta$ and some $t_1, t_2 > t_0$ such that

$$\| x(t_1, t_0, x_0) \| = \delta_2 \quad , \quad \| x(t_2, t_0, x_0) \| = \varepsilon$$

and

$$\delta_2 \leq \| x(t, t_0, x_0) \| \leq \varepsilon \quad , \quad t \in [t_1, t_2] . \tag{10}$$

Let $\delta_2 = \tau > 0$ so there exists a $V_{2,\tau}$ satisfying hypothesis (ii).

Let $\eta_0 = \min\{\eta_1, \eta\}$, $L_0 = \max\{L, M\}$ where L, and M are the Lipschitz constants of V_1 and $V_{2,\tau}$ respectively, and θ be an $\frac{\eta_0}{2 L_0}$ translation number for $f(t, x)$ such that $t_0 + \theta > 0$. Clearly for $t \in (-\infty, \infty)$

$$\| f(t + \theta, x) - f(t, x) \| < \frac{\eta_0}{2 L_0} \tag{11}$$

if $x \in S$ any compact set in $S(\rho)$.

Let

$$m(t) = V_1(t + \theta, x(t, t_0, x_0)) + V_{2,\tau}(t + \theta, x(t, t_0, x_0)) \qquad t \in [t_1, t_2]. \tag{12}$$

For $h > 0$,

$$\frac{m(t+h) - m(t)}{h} =$$

$$= \frac{V_1(t + h + \theta, x(t + h, t_0, x_0)) + V_{2,\tau}(t + h + \theta, x(t + h, t_0, x))}{h}$$

$$- \frac{V_1(t + \theta, x(t, t_0, x_0)) - V_{2,\tau}(t + \theta, x(t, t_0, x_0))}{h} .$$

$$\limsup_{h\to 0} \frac{V_1(t+h+\theta, x(t, +h, t_0, x_0)) - V_1(t+\theta, x(t, t_0, x_0))}{h}$$

$$\leq \lim_{h\to 0} \frac{\|\varepsilon\|}{h} + L\,\|f(t+\theta, x(t)) - f(t, x(t))\| + D^+ V_1(t+\theta, x(t))$$

where $\frac{\|\varepsilon\|}{h} \to 0$ as $h \to 0$.

Similarly,

$$\limsup_{h\to 0} \frac{V_{2,\tau}(t+h+\theta, x(t+h, t_0, x_0)) - V_{2,\tau}(t+\theta, x(t))}{h}$$

$$\leq M\,\|f(t, +\theta, x(t)) - f(t, x(t))\| + D^+ V_{2,\tau}(t+\theta, x(t)).$$

Hence

$$\begin{aligned} D^+ m(t) &\leq L\,\|f(t+\theta, x(t)) - f(t, x(t))\| + M\,\|f(t+\theta, x(t)) \\ &\quad - f(t, x(t)\| + D^+ V_{2,\tau}(t+\theta, x(t)) + D^+ V_1(t+\theta, x(t)) \\ &\leq g_2(t+\theta, V_1(t+\theta, x(t)) + V_{2,\tau}(t+\theta, x(t)) \\ &\quad + 2\,L_0\,\|f(t+\theta, x(t)) - f(t, x(t)\| \leq g_2(t+\theta, m(t)) + \eta_0 \\ &\leq g_2(t+\theta, m(t)) + \eta \qquad t \in [t_1, t_2]. \end{aligned}$$

Let $\bar{t} = \min\{t_1^*, t_2^*\}$ and choose $\tau_0 + \theta = t_1$, and $\bar{t} = t_2 + \theta$, then an application of (ii) and Theorem 1.4.1 of [2] implies that

$$m(t) \leq r_2(t+\theta, t_1, V_1(t_1, x(t_1, t_0, x_0)) + V_{2,\tau}(t_1, x(t_1, t_0, x_0))\,\eta)$$

where $r_2(t+\theta, t_1, w_0, \eta)$ is the maximum solution of (5) such that

$$r_2(t_1, t_1, w_0, \eta) = w_0.$$

Also by (i), and Theorem 1.4.1 of [2], we have

$$V_1(t_1, x(t_1, t_0, x_0)) \leq r_1(t_1, t_0, V_1(t_0, x_0), \eta_1)$$

where $r_1(t, t_0, u_0, \eta_1)$ is the maximal solution of (8). By (7) and (9), we obtain

$$V_1(t_1, x(t_1, t_0, x_0)) < \frac{\delta_0}{2}. \tag{13}$$

By the assumption on $V_{2,\tau}$, (6) and (10),

$$V_{2,\tau}(t_1, x(t_1, t_0, x_0)) \leq a(\delta_2) < \frac{\delta_0}{2}. \tag{14}$$

Thus, since $V_1(t, x) \geq 0$,

$$b(\varepsilon) = b(\| x(t_2, t_0, x_0) \|) \leq V_{2,\tau}(t_2 + \theta, x(t_2)),$$

and

$$b(\varepsilon) \leq V_{2,\tau}(t_2 + \theta, x(t_2)) \leq V_1(t_2 + \theta, x(t_2)) + V_{2,\tau}(t_2 + \theta, x(t_2))$$
$$\leq r_2(t_2 + \theta, t_0, V_1(t_1, x(t_1)) + V_{2,\tau}(t_1, x(t_1)), \eta).$$

Now by (13) and (14), $m(t_1) < \delta$, hence

$$b(\varepsilon) \leq r_2(t_2 + \theta, t_1, V_1(t_1 x(t_1)) + V_{2,\tau}(t_1, x(t_1), \eta) < b(\varepsilon),$$

by (4), which is a contradiction. The theorem is therefore established.

THEOREM 2.4. *Suppose the trivial solution $u = 0$ of* (2) *is strongly equi-asymptotically stable and that assumptions* (i), (ii) *and* (iii) *of Theorem 2.3 hold. Then, if the trivial solution $w = 0$ of*

(15) $$w^1 = g_2(t, w) \quad , \quad w(t_0) = w_0 \geq 0$$

is strongly uniformly stable, then the trivial solution of (1) *is perfectly equi-asymptotically stable.*

Proof. By Theorem 2.3, the trivial solution of (1) is perfectly equistable. We now show that it is equi-asymptotically stable. Let $0 < \varepsilon < \rho$, and $t_0 \in (-\infty, \infty)$. Given $b(\varepsilon) > 0$ and $\tau_0 \in R^+$ $\exists \delta_0 = \delta_0(\varepsilon), \eta = \eta(\varepsilon)$ and a compact set $K = [\tau_0, t_1^*]$ such that if $w_0 < \delta_0$, then

$$w(t, \tau_0, w_0, \eta) < b(\varepsilon), \qquad t \in [\tau_0, t_1^*]$$

where $w(t, \tau_0, w_0, \eta)$ is any solution of (15). With this δ_0, we can schoose $\delta_2 = \delta_2(\varepsilon)$ such that

(16) $$a(\delta_2) < \frac{\delta_0}{2},$$

and since $u = 0$ is strongly equi-asymptotically stable, given $\frac{\delta_0}{2}$ and $\tau_0 \in R^+$ and any compact set $[\tau_0, t_2^*]$, $\exists \delta^* = \delta^*(\tau_0), \eta_1 = \eta_1(\varepsilon)$ and $T(\tau_0, \varepsilon)$ such that

(17) $$u(t, \tau_0, u_0, \eta_1) < \frac{\delta_0}{2}, \qquad t \geq t_0 + T$$

where $u(t, \tau_0, u_0, \eta_1)$ is any solution of equation (8), provided $u_0 < \delta^*$. Choose $u_0 = V_1(t_0, x_0)$. Since $V_1(t, x)$ is continuous and $V_1(t, 0) \equiv 0$, $\exists \delta_1 = \delta_1(\tau_0, \rho)$ such that

$$\| x_0 \| < \delta_1 \quad \text{and} \quad V_1(t_0, x_0) < \delta^* = \delta^*(\tau_0)$$

hold together

Set $\delta_0^* = \min\{\delta_1, \delta_2\}$, then

$$\| x(t, t_0, x_0) \| < \varepsilon \qquad \text{for} \quad t \geq \tau_0 + T,$$

provided $\| x_0 \| < \delta_0^*$.

Choose θ as in Theorem 2.3 and define

$$m(t) = V_1(t + \theta, x(t, t_0, x_0)) + V_{2,\tau}(t + \theta, x(t, t_0, x_0)) \qquad t \geq \tau_0 + T,$$

where $\delta_2 = \tau$, and $x(t, t_0, x_0)$ is any solution of (1) such that $\| x_0 \| < \delta_0^*$.

Then proceeding as in Theorem 2.3,

$$D^+ m(t) \leq g_2(t + \theta, m(t) + \eta \qquad \text{for} \quad t \geq \tau_0 + T.$$

Choosing θ such that $\tau_0 = t_0 + \theta$ and using the assumptions, and Theorem 1.4.1 of [2],

$$V_1(t + \theta, x(t)) + V_{2,\tau}(t + \theta, x(t)) \leq r_2(t + \theta, \tau_0, V_1 + V_{2,\tau}, \eta) \quad t \geq \tau_0 + T.$$

Since $V_1(t, x) \geq 0$, and by the assumption on $V_{2,\tau}$,

$$b(\| x(t) \|) \leq V_{2,\tau}(t + \theta, x(t)) \leq V_1(t + \theta, x(t)) + V_{2,\tau}(t + \theta, x(t))$$
$$\leq r_2(t + \theta, \tau_0, V_1(t + \theta, x(t)) + V_{2,\tau}(t + \theta, x(t)), \eta).$$

Now using condition (16), (17) and the assumptions on $V_{2,\tau}$,

$$V_1(t, x(t, t_0, x_0)) + V_{2,\tau}(t, x(t, t_0, x_0)) < \delta_0$$

and hence, by (17),

$$b(\| x(t) \|) \leq r_2(t + \theta, \tau_0, V_1, + V_{2,\tau}, \eta) < b(\varepsilon) \qquad \text{for} \quad t \geq \tau_0 + T.$$

Hence $\| x(t) \| < \varepsilon$ for $t \geq \tau_0 + T$ provided $\| x_0 \| < \delta_0^*$.

THEOREM 2.5. *Suppose that assumptions* (*i*), (*ii*), (*iii*), (*iv*) *of Theorem 2.3 hold. Let f of equation* (1) *be such that* $f \in C([0, \infty) \times S(\rho), R^n)$ *and* $f(t, 0) = 0$. *Then the strong equistability of the trivial solution* $u = 0$ *of equation* (2) *and the strong uniform stability of the trivial solution* $w = 0$ *of equation* (3) *imply the strong equi-stability of the trivial solution* $x = 0$ *of equation* (1).

Proof. By the assumptions on the solutions $u = 0$ and $w = 0$ and the same procedure of Theorem 2.3, given $\varepsilon > 0$, $t_0 \in [0, \infty)$, and choosing

$$u_0 = V_1(t_0, x_0) \quad , \quad \exists \delta^* = \delta^*(t_0, \varepsilon) \qquad \text{and} \quad \delta_1 = \delta_1(t, \varepsilon)$$

such that

$$\| x_0 \| < \delta_1 \qquad \text{and} \quad V_1(t_0, x_0) < \delta^* \qquad \text{hold together.}$$

Set $\delta = \min\{\delta_1, \delta_2\}$ and let $\eta_0 = \min\{\eta_1, \eta\}$, $L_0 = \max\{L, M\}$, then the claim is that with δ and η_0 the trivial solution (1) is strongly equistable with $K = [t_0, \bar{t}]$ where $\bar{t} = \min\{t_1^*, t_2^*\}$ and $t_0 = \tau_0$. If not, proceed in the same way as in Theorem 2.3 and set

$$m(t) = V_1(t, x(t, t_0, x_0)) + V_{2,\delta_2}(t, x(t, t_0, x_0)) \qquad t \in [t_1, t_2],$$

whence

$$D^+ m(t) \leq g_2(t, m(t)) + \eta,$$

and the remaining part the arguments proceeds as in Theorem 2.3, to obtain the desired result.

REMARKS. Theorems 2.3, 2.4 and 2.5 improve significantly, the perfect equi-stability, the perfect equi-asymptotic stability and the strong equistability results in [2, § 3.18] respectively.

REFERENCES

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