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Analisi funzionale. — *Convergence of sequence of iterates of generalized contractions.* Nota di KANHAYA L. SINGH, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si dà un'estensione di un risultato contenuto in una precedente Nota lincea [11] riguardante la convergenza di una successione di iterate di contrazioni generalizzate in uno spazio di Banach strettamente convesso.

In a recent paper S. P. Singh [11] proved the following theorem: Let C be a closed, bounded, convex subset of a strictly convex Banach space; let $T : C \to C$ be a densifying mapping satisfying a certain condition (A); then, for x in C, the sequence of iterates $\{T_\lambda^n x_0\}$, where $T : C \to C$ is the mapping defined by $T_\lambda = \lambda T + (1 - \lambda) I$, $0 < \lambda < 1$, converges to a fixed point of T. In the present Note we prove a theorem for the convergence of sequence of iterates of generalized contraction mappings which are more general than those of S. P. Singh.

A sequence of iterates $\{T^n x_0\}$ of a non expansive mapping T need not converge to a fixed point of T. Therefore one considers a mapping of the form

$$T_\lambda = \lambda T + (1 - \lambda) I , \qquad 0 < \lambda < 1$$

and then the sequence of iterates $\{T_\lambda^n x_0\}$, under suitable restrictions, converges to a fixed point of T. Before we state and prove our theorem we need to recall the following.

DEFINITION 1.1. Let X be a Banach space. A mapping $T : X \to X$ is said to *satisfy condition (A)* if for all x , y in X we have

$$\| Tx - Ty \| \leq a_1 \| x - y \| + a_2 (\| x - Tx \| + \| y - Ty \|) + a_3 (\| y - Tx \| + \| x - Ty \|)$$

where $a_i \geq 0 , i = 1 , 2 , 3$ and $a_1 + 2 a_2 + 2 a_3 \leq 1$.

DEFINITION 1.2. Let X be a Banach space. A mapping $T : X \to X$ is said to be a *generalized contraction* if

$$\text{(B)} \qquad \| Tx - Ty \| \leq \max \{ \| x - y \| , \| x - Ty \| , \| y - Ty , (\| x - Tx \| + \| y - Ty \|)/2 , (\| x - Ty \| + \| y - Tx \|)/2\}$$

for all x , y in X.

(*) Nella seduta del 12 febbraio 1977.

DEFINITION 1.3. Let X be a Banach space. A mapping $T : X \to X$ is said to have property (C) is for all x, y in X we have

$$\| Tx - Ty \| \leq \max \{ \| x - y \| , (\| x - Tx \| + \| y - Ty \|)/2 , (\| x - Ty \| + \| y - Tx \|)/2 \} .$$

It is clear that condition (A) and condition (C) imply condition (B). The following example shows that a generalized contraction need not satisfy either condition (A) or condition (C).

Example 1.1. Let

$$M_1 = \{ m/n : m = 0 , 1 , 3 , 9 , \cdots ; \quad n = 1 , 4 , \cdots , 3k + 1 , \cdots \} ,$$

$$M_2 = \{ m/n : m = 1 , 3 , 9 , 27 , \cdots ; \ n = 2 , 5 , \cdots , 3k + 2 , \cdots \} ,$$

and let $M = M_1 \cup M_2$ with the usual metric. Define $T : M \to M$ by

$$Tx = \begin{cases} 3x/5 & \text{for } x \text{ in } M_1 \\ x/8 & \text{for } x \text{ in } M_2 . \end{cases}$$

The mapping T is a generalized contraction. Indeed, if both x and y are in M_1 or in M_2, then $\| Tx - Ty \| \leq 3 \| x - y \|/5 < \| x - y \|$. Now let x be for example in M_1 and y in M_2. Then

$$x > (5/24)\, y \text{ implies } | Tx - Ty | = 3/5\, (x - (5/24)\, y) \leq 3/5\, (x - (1/8)\, y) = = 3/5\, | x - Ty | ;$$

$$x < (5/24)\, y \text{ implies } | Tx - Ty | = 3/5\, ((5/24)\, y - x) < 3/5\, (y - x) = = 3/5\, | y - x | .$$

Therefore, T on M satisfies condition (B).

To show that T does not satisfy neither condition (A) nor (C), let $x = 1$ and $y = 1/2$. Then we have

$$| Tx - Ty | = 3/5 - 1/16 = 43/80 .$$

However

$$a_1 \| x - y \| + a_2 (\| x - Tx \| + \| y - Ty \|) + a_3 (\| x - Ty \| + \| y - Tx \|)$$
$$= a_1/2 + a_2 (2/5 + 7/16) + a_3 (15/16 + 1/10) = a_1/2 + a_2 \cdot 67/80 + a_3\, 83/80$$
$$< (a_1 + 2\, a_2 + 2\, a_3)\, 83/160 \leq 83/160 < 43/80 = | Tx - Ty | .$$

Hence condition (A) is not satisfied. Moreover,

$$\max \{ \| x - y \| , (\| x - Tx \| + \| y - Ty \|)/2 , (\| x - Ty \| + \| y - Tx \|)/2 \}$$
$$= \max \{ 1/2 , (2/5 + 7/16) , (15/16 + 1/10)/2 \}$$
$$= \max \{ 1/2 , 67/160 , 83/160 \} = 83/160 < | Tx - Ty | .$$

Thus condition (C) is not satisfied.

DEFINITION 1.4. Let X be a real Banach space and D a bounded subset of X. The *measure of non compactness* of D, denoted by $\alpha(D)$, is defined as follows

$\alpha(D) = \inf \{\varepsilon > 0 : D$ can be covered by a finite number of subsets of diameter $< \varepsilon\}$.

$\alpha(D)$ has the following properties:

1) $0 \leq \alpha(D) \leq \delta(D)$, where $\delta(D)$ is the diameter of D;

2) $\alpha(D) = 0$ if and only if D is precompact (i.e., $\bar{D}$ is compact);

3) $\alpha(D) = \alpha(\bar{D})$, where $\bar{D}$ is the closure of D;

4) $\alpha(C \cup D) = \max \{\alpha(C), \alpha(D)\}$;

5) $\alpha(C + D) \leq \alpha(C) + \alpha(D)$, where $C + D = \{c + d : c$ in C and d in D$\}$;

6) $\alpha(\beta D) = |\beta| \alpha(D)$, where β is any real number;

7) $C \subset D$ implies $\alpha(C) \leq \alpha(D)$;

8) $\alpha(S(B, r)) \leq \alpha(B) + 2r$, where $S(B, r) = \{x$ in $X : d(x, B) < r\}$.

Closely related to the notion of measure of non compactness is the concept of k-set contraction introduced by Darbo [2].

DEFINITION 1.5. (*Darbo*). Let X be a real Banach space. A continuous mapping $T : X \rightarrow X$ is said to be *k-set contraction* if for any bounded subset D of X we have

$$\alpha(T(D)) \leq k\alpha(D).$$

DEFINITION 1.6. (*Furi and Vignoli*). Let X be a real Banach space. A continuous mapping $T : X \rightarrow X$ is said to be densifying if for any bounded subset D of X such that $\alpha(D) \neq 0$ we have

$$\alpha(T(D)) < \alpha(D).$$

We will need the following theorems for the proof of our theorem.

THEOREM A. (*Furi and Vignoli* [4]). *Let* X *be a Banach space and* C *a closed, bounded, convex subset of* X. *Let* $T : C \rightarrow C$ *be a densifying mapping. Then* T *has at least one fixed point in* C.

THEOREM B (*Diaz and Metcalf* [3]). *Let* $T : X \rightarrow X$ *be a continuous mapping. Suppose*

(1) $F(T) \neq \phi$, *where* $F(T)$ *is the set of fixed points of* T;

(2) *for each* y *in* X, y *not in* F(T) *and each* u *in* F(T)

$$\| Ty - u \| < \| y - u \| ;$$

let x_0 *in* X. *Then either* $\{T^n x_0\}$ *has no convergent subsequence or* $\{T^n x_0\}$ *converges to a fixed point of* T.

We prove the following

THEOREM. *Let* X *be a strictly convex Banach space and* C *a closed, bounded, convex subset of* X. *Let* $T : C \to C$ *be a generalized contractive densifying mapping. Then, for any* x_0 *in* C, *the sequence of iterates* $\{T_\lambda^n x_0\}$, *where* $T_\lambda : C \to C$ *is a mapping defined by* $T_\lambda = \lambda T + (1 - \lambda) I$, $0 < \lambda < 1$; *converges to a fixed point of* T.

Proof. Since T is densifying on C, T_λ is also densifying. Indeed, the continuity of T_λ follows from that of T. It remains to show that for any bounded subset B of C such that $\alpha(B) \neq 0$ we must have

$$\alpha(T_\lambda(B)) < \alpha(B) .$$

By definition of T_λ we have

$$T_\lambda(B) \subset \lambda T(B) + (1 - \lambda) B$$

so

$\alpha(T_\lambda(B)) \leq \lambda\alpha(T(B)) + (1 - \lambda)\alpha(B) < \alpha(B) + (1 - \lambda)\alpha(B)$ (since T is densifying) $= \alpha(B)$.

Since T is densifying, we conclude from Theorem A that $F(T) \neq \phi$. Moreover $F(T_\lambda) \neq \phi$, since $F(T) = F(T_\lambda)$. Let $D = \bigcup_{n=0}^{\infty} T_\lambda^n x_0$; then $T_\lambda(D) = \bigcup_{n=1}^{\infty} T_\lambda^n x_0$. Thus $T_\lambda(D) \subset D$, and $D = \{x_0\} \cup T_\lambda(D)$.

Let $\bar{D}$ be the closure of D; then $T_\lambda(\bar{D}) \subset \overline{T_\lambda(D)} \subset \bar{D}$, i.e., $\bar{D}$ is invariant under T_λ.

We now show that $\bar{D}$ is compact. By property (2) of α it is enough to show that $\alpha(\bar{D}) = 0$. Using property (3) it suffices to show that $\alpha(D) = 0$. Suppose not, i.e. $\alpha(D) > 0$. Since $D = \{x_0\} \cup T_\lambda(D)$,

$\alpha(D) = \max\{\alpha(x_0), \alpha(T_\lambda(D))\}$ (by property (4)) $= \max\{0, \alpha(T_\lambda(D))\} = \alpha(T_\lambda(D))$, a contradiction to the fact that T_λ is densifying Hence, $\alpha(D) = 0$.

Thus $\bar{D}$ is compact. Hence the sequence of iterates $\{T_\lambda^n x_0\}$ has a convergent subsequence.

In order to apply Theorem B, we need to show that for each $x \notin F(T_\lambda)$ and u in $F(T_\lambda)$ (hence $x \notin F(T)$ and u in $F(T)$), we have

$$\| T_\lambda x - u \| < \| x - u \| .$$

Since T is a generalized contraction we have

(a) $\| Tx - u \| = \| Tx - Tu \| \leq \max \{\| x - u \| \, , \; \| x - Tu \| \, , \; \| u - Tu \|$
$(\| x - Tx \| + \| u - Tu \|)/2 \, , \; (\| x - Tu \| + \| u - Tx \|)/2\}$
$= \max \{\| x - u \| \, , \; \| x - u \| \, , \; \| x - Tx \|/2 \, , \; (\| x - u \| +$
$+ \| u - Tx \|)/2\}$.

Now, $\| Tx - u \| \leq \| x - Tx \|)/2 \leq (\| x - u \| + \| u - Tx \|)/2$ implies

$$\| Tx - u \| \leq \| x - u \| .$$

Thus we can write (a) as

(b) $$\| Tx - u \| \leq \| x - u \|.$$

Since u belongs to F (T) and x does not belong to F (T), it follows that $x \neq u$.

Now

$$\| T_\lambda x - u \| = \| T_\lambda x - T_\lambda u \| = \| \lambda (Tx - u) + (1 - \lambda)(x - u) \|$$
$$= \| x - u \| \, \| \lambda (Tx - u)/\| x - u \| + (1 - \lambda)(x - u)/\| x - u \| \, \|.$$

If strict inequality holds in (b), we obtain

$$\| T_\lambda x - u \| \leq \{\lambda \| Tx - u \|/\| x - u \| + (1 - \lambda) \| x - u \|/\| x - u \|\} \, \| x - u \|$$
$$< \{\lambda + (1 - \lambda)\} \, \| x - u \| = \| x - u \| .$$

However, if equality sign holds in (b), then since $(Tx - u)/\| x - u \|$, and $(x - u)/\| x - u \|$ have unit norm and X is strictly convex, we get

$$\| \lambda (Tx - u)/\| x - u \| + \| (1 - \lambda)(x - u)/\| x - u \| \, \| < 1 .$$

Thus $\| T_\lambda x - u \| < \| x - u \|$, whence the proof.

As a corollary of our Theorem we have the following result of Massabo [6].

COROLLARY. *Let* $T : K \to K$ *be a densifying mapping defined on a closed, convex, bounded subset* K *of a strictly convex Banach space. Let* T *satisfy condition* (C). *Then, for each* x *in* K, *the Picard–Banach sequence starting from* x *and generated by the mapping* T_λ

$$T_\lambda (x) = \lambda T (x) + (1 - \lambda) x \, , \qquad 0 < \lambda < 1$$

converges to a fixed point of T.

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