Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Common fixed, point theorems on metric spaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **62** (1977), n.2, p. 150–153.

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Analisi matematica. — *Common fixed point theorems on metric spaces.* Nota di BRIAN FISHER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si dimostra che, se S ed T sono applicazioni di uno spazio metrico completo limitato X in se, tale che

$$d(Sx, Ty) \leq c \max\{d(x, Ty), d(y, Sx)\} \qquad (0 \leq c < 1)$$

per tutti gli x, y di X, allora S ed T ammettono un unico punto fisso comune.

We first of all prove the following theorem:

THEOREM 1. *Suppose* S *and* T *are mappings of the metric space* X *into itself satisfying the inequality*

$$d(Sx, Ty) \leq c \max\{d(x, Ty), d(y, Sx)\}$$

for all x, y *in* X, *where* $0 \leq c < 1$. *Suppose further that* S *has a fixed point* z. *Then* z *is a unique common fixed point of* S *and* T.

Proof. We have

$$d(z, Tz) = d(Sz, Tz) \leq c \max\{d(z, Tz), d(z, Sz)\} = cd(z, Tz).$$

Since $c < 1$, it follows that z is also a fixed point of T.

Let us now suppose that w is a second fixed point of S. By what we have just proved w is also a fixed point of T and so

$$d(z, w) = d(Sz, Tw) \leq c \max\{d(z, Tw), d(w, Sz)\} = cd(z, w).$$

The uniqueness of the common fixed point follows immediately.

By noting that

$$d(x, Ty) + d(y, Sx) \leq 2 \max\{d(x, Ty), d(y, Sx)\}$$

we have the following

THEOREM 2. *Suppose* S *and* T *are mappings of the metric space* X *into itself satisfying the inequality*

$$d(Sx, Ty) \leq c\{d(x, Ty) + d(y, Sx)\}$$

for all x, y *in* X, *where* $0 \leq c < \frac{1}{2}$. *Suppose further that* S *has a fixed point* z. *Then* z *is a unique common fixed point of* S *and* T.

(*) Nella seduta del 12 febbraio 1977.

We now prove the following

THEOREM 3. *Suppose* S *and* T *are mappings of the complete and bounded metric space* X *into itself satisfying the inequality*

$$d(Sx, Ty) \leq c \max\{d(x, Ty), d(y, Sx)\}$$

for all x, y *in* X, *where* $0 \leq c < 1$. *Then* S *and* T *have a unique common fixed point* z.

Proof. Let x be an arbitrary point in X. Then

$$d(S^n x, T^r x) \leq c \max\{d(S^{n-1} x, T^r x), d(T^{r-1} x, S^n x)\} \leq$$
$$\leq c^i \max\{d(S^{n-j} x, T^{r-i+j} x) : j = 0, 1, \cdots, i\}$$

for $n, r \geq i$. Since X is bounded

$$M = \sup\{d(x, y) : x, y \in X\} < \infty .$$

For arbitrary $\varepsilon > 0$, choose N so that

$$c^N M < \varepsilon .$$

It follows that

$$d(S^n x, T^r x) < \varepsilon$$

for $n, r \geq N$ and so

$$d(S^n x, S^m x) \leq d(S^n x, T^r x) + d(T^r x, S^m x) < 2\varepsilon$$

for $m, n, r \geq N$. Hence $\{S^n x\}$ is a Cauchy sequence in the complete metric space X and so has a limit z in X. Similarly, $\{T^n x\}$ is a Cauchy sequence in X and since

$$d(S^n x, T^n x) < \varepsilon$$

for $n > N$, the sequence $\{T^n x\}$ also converges to z.

We now have

$$d(z, Sz) \leq d(z, T^n x) + d(T^n x, Sz) \leq d(z, T^n x) +$$
$$+ c \max\{d(T^{n-1} x, Sz), d(z, T^n x)\}$$

and on letting n tend to infinity we see that

$$d(z, Sz) \leq c d(z, Sz) .$$

Since $c < 1$, it follows that z is a fixed point of S.

That z is a unique common fixed point of S and T, now follows from Theorem 1.

The next theorem follows immediately:

THEOREM 4. *Suppose* S *and* T *are mappings of the complete and bounded metric space* X *into itself satisfying the inequality*

$$d(\mathrm{S}x, \mathrm{T}y) \leq c\{d(x, \mathrm{T}y) + d(y, \mathrm{S}x)\}$$

for all x, y *in* X, *where* $0 \leq c < \frac{1}{2}$. *Then* S *and* T *have a unique common fixed point* z.

We finally prove two theorems for compact metric spaces. First of all we have

THEOREM 5. *Suppose* S *and* T *are continuous mappings of the compact metric space* X *into itself satisfying either the inequality*

$$d(\mathrm{S}x, \mathrm{T}y) < \max\{d(x, \mathrm{T}y), d(y, \mathrm{S}x)\}, \qquad \text{if} \quad \max\{d(x, \mathrm{T}y), d(y, \mathrm{S}x)\} \neq 0$$

or the equality

$$d(\mathrm{S}x, \mathrm{T}y) = 0, \qquad \text{if} \quad \max\{d(x, \mathrm{T}y), d(y, \mathrm{S}x)\} = 0$$

for all x, y *in* X. *Then* S *and* T *have a unique fixed point* z.

Proof. Suppose first of all that there exists $c < 1$ such that

$$d(\mathrm{S}x, \mathrm{T}y) \leq c \max\{d(x, \mathrm{T}y), d(y, \mathrm{S}x)\}$$

for all x, y in X. The result then follows from Theorem 3, since a compact metric space is bounded.

If no such c exists, we can find a sequence of positive real numbers $\{c_n\}$ converting to zero and sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$d(\mathrm{S}x_n, \mathrm{T}y_n) > (1 - c_n) \max\{d(x_n, \mathrm{T}y_n), d(y_n, \mathrm{S}x_n)\}$$

for $n = 1, 2, \cdots$. Since X is compact we can find convergent subsequences $\{x_{n(r)}\}$ and $\{y_{n(r)}\}$ of $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively. We then have

$$d(\mathrm{S}x_{n(r)}, \mathrm{T}y_{n(r)}) > (1 - c_{n(r)}) \max\{d(x_{n(r)}, \mathrm{T}y_{n(r)}), d(y_{n(r)}, \mathrm{S}x_{n(r)})\}$$

and, on letting r tend to infinity, we see that since S and T are continuous

$$d(\mathrm{S}x, \mathrm{T}y) \geq \max\{d(x, \mathrm{T}y), d(y, \mathrm{S}x)\},$$

giving a contradiction unless

$$x = y = \mathrm{S}x = \mathrm{T}y.$$

Putting $x = y = z$, it follows that z is a common fixed point of S and T.

Now suppose that S and T have a second common fixed point w. Then

$$d(z, w) = d(\mathrm{S}z, \mathrm{T}w) < \max\{d(z, \mathrm{T}w), d(w, \mathrm{S}z)\} = d(z, w),$$

giving a contradiction unless $z = w$. The common fixed point is therefore unique.

The final theorem follows immediately:

THEOREM 6. *Suppose* S *and* T *are continuous mappings of the campact metric space* X *into itself satisfying either the inequality*

$$d(\mathrm{S}x , \mathrm{T}y) < \tfrac{1}{2}\{d(x , \mathrm{T}y) + d(y , \mathrm{S}x)\} , \qquad \textit{if} \quad d(x , \mathrm{T}y) + d(y , \mathrm{S}x) \neq 0$$

or the equality

$$d(\mathrm{S}x , \mathrm{T}y) = 0 , \qquad \textit{if} \quad d(x , \mathrm{T}y) + d(y , \mathrm{S}x) = 0$$

for all x , y *in* X. *Then* S *and* T *have a unique common fixed point* z.

On putting S = T in Theorems 3, 4, 5 and 6 we get four special cases. The condition that X be bounded is then not needed in Theorem 4, see [1].

REFERENCE

[1] B. FISHER – *A fixed point theorem*, «Mathematics Magazine», *48*, 223–5.